CS 381 Fall 2000
Solutions to Homework 2

Handout #4
Problem 1

DFA 1

DFA 2

UNION OF DFAs 1 & 2
In case of intersection (2,3) is the only final state.

Intersection of DFAs 1 & 2
In union, all the states are final states
Problem 2

DFA accepting $A_{5,2}$

Proof that the DFA accepts $A_{5,2}$

Claim: If $w = a_1a_2...a_n$ and $\text{decimal}(w) = 5k + r$, $0 \leq r < 5$, where $\text{decimal}(w)$ is the decimal value represented by the binary string, then the above DFA after reading $w$ is in state $r$.

Proof of the claim by induction on length of $w$: If $|w| = 1$, then it obvious that the claim is true. Assume that it is true for $|w| < n$. Consider $w = a_1a_2...a_n$. Let $w' = a_1a_2...a_{n-1}$. Then $\text{decimal}(w) = 2 \times \text{decimal}(w') + a_n$. If $\text{decimal}(w') = 5k + r$, then $\text{decimal}(w) = 2 \times (5k + r) + a_n = 5 \times 2k + 2r + a_n$. Hence $w$ should be in state $2r + a_n (mod 5)$. By induction hypothesis, after reading $w'$, the DFA will be in state $r$. Check that for all possible values of $r (0, 1, 2, 3, 4)$ and all possible $a_n (0, 1)$, the DFA goes to the state $2r + a_n (mod 5)$. Hence the claim.

Hence using this claim, it is obvious the the DFA accepts $A_{5,2}$, because if $w \in A_{5,2}$, then the DFA after reading $w$ will be in state 0 which is the final state.

Handout #5

Problem 1

Construction of the new DFA $M'$: Let $s$ be the start state of $M$. The new DFA $M'$ has all the states of $M$ plus the additional state $s'$. If $s$ is a final state in $M$, then $s'$ is a final state in $M'$. The start state of $M'$ is again $s$. The transition function $\delta'$ of $M'$ is:

$\forall$ states $q$ in $M$, if $\delta(q, a) = q'$ and $q' \neq s$, then $\delta'(q, a) = q'$ else $\delta'(q, a) = s'$.

Finally if $\delta(s, a) \neq s$, then $\delta'(s', a) = \delta(s, a)$ else $\delta'(s', a) = s'$.

So from the construction it is obvious that in $M'$, the start state $s$ has no incoming arrows.

Claim: If $M$ reads a string $w$ and reaches state $q$, then $M'$ on $w$ will reach $q$ if $q \neq s$ else $s'$.

An immediate inference of this claim is if $M'$ reaches a final state iff $M$ reaches a final state on the same input, i.e $M$ and $M'$ are equivalent.

Proof by induction on the length of $w$: If $|w| = 1$, then easy to see that the above claim is true. Therefore let it be true for $|w| < n$. Consider $|w| = n$. Let $w = a_1a_2...a_n$. Let $w' = a_1a_2...a_{n-1}$. $\delta'(s, w) = \delta'(\delta'(s, w'), a_n)$ Let $\delta'(s, w') = q_{n-1}$. There are 2 cases: $q_{n-1} = s'$ and $q_{n-1} \neq s'$. Consider the case $q_{n-1} \neq s'$. Then by induction hypothesis, $\delta(s, w') = q_{n-1}$. Let $\delta(q_{n-1}, a_n) = q_n$. If $q_n \neq s$, then by definition $\delta'(q_{n-1}, a_n) = q_n$ and we are done. If $q_n = s$, then by definition $\delta'(q_{n-1}, a_n) = s'$ and again we are done. The other case $q_{n-1} = s'$ is analysed similarly. Hence the claim.
**Problem 3**

**The idea**: A regular $\Rightarrow \exists$ DFA $M$, such that language accepted by $M$ is $A$. From $M$, we will construct NFA $M^R$ such that language accepted it is $A^R$, the reverse of $A$. This implies $M^R$ is regular.

**Construction of $M^R$**: Let $s$ be the start state of $M$. $M^R$ includes all the states of $M$ plus an additional state $t$. The transition function $\delta'$ of $M^R$ is defined as follows:

- $\forall$ states $q$ in $M$, $\delta'(q,a)=$ includes $r$ iff $\delta(r,a) = q$ (reversing the edges in $M$)
- $\delta'(t,a) = f \quad \forall f$ which are final states of $M$

$t$ is the start state of $M^R$ and $s$ the final state.

**Claim**: Language accepted by $M^R$ is $A^R$.

**Proof**: Suppose $w = a_1a_2...a_n$ is accepted by $M$. Let $M$ go reach the states $q_1, q_2, \ldots, q_n$ after reading $a_1, a_1a_2, \ldots, a_1a_2a_3 \ldots a_n$ respectively. $q_n$ is a final state. Then in $M^R$ there is a path from $t$ to $s$ via $q_n, q_{n-1}, \ldots, q_1$ traversing the edges $a_n, a_{n-1}, \ldots, a_1$. Hence $w^R$ is accepted by $M^R$.

Now suppose $w = a_1a_2...a_n$ is not accepted by $M$. We have to show that $w^R$ is not accepted by $M^R$. On the contrary assume that $w^R$ is accepted by $M^R$. This implies that starting from $t$ we can reach $s$ through a path consisting of edges $a_n, a_{n-1}, \ldots, a_1$. From our construction of $M^R$, it follows that there is a path in $M$ from $s$ to a final state consisting of edges $a_1, a_2, \ldots, a_n$, which implies $w$ is accepted by $M$, a contradiction. Hence $w^R$ is not accepted by $M^R$. Hence the claim.