CS 3220: SVD WORKSHEET

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Notation

This worksheet focuses on the SVD and what information we can extract from it. Consequently, we will need some standardized notation to talk about the SVD. For the purposes of this discussion we will only consider square matrices A that are $n \times n$. We denote the SVD of A as

$$A = U\Sigma V^T,$$

where U and V are $n \times n$ orthogonal matrices and Σ is a diagonal matrix with non-negative entries. More specifically, we let

$$U = \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix}, V = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix},$$

where by convention $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Lastly, we let r denote the rank of A. If A is rank r then $\sigma_i = 0$ for $i > r$.

Fundamental subspaces

We discussed the four fundamental subspaces in class a few lectures ago. Now we will see how the SVD "reveals" several facts about them.

- For each of the four fundamental subspaces find an orthonormal basis for the subspace using the elements of the SVD.
- Now, argue that the kernel and co-range are orthogonal, and that the range and co-kernel are orthgonal.

Note that the first part of this question gives us a way to compute projectors onto the various subspaces.

How much A can shrink vectors

We discussed in class that we can relate the 2-norm of A to its SVD. However, we also previously discussed how we may also be interested in understanding when a formally full rank matrix A can map vectors to small vectors (more specifically, we are interested in this property when the norm of A is not also small; more on this point later). As with the 2-norm, singular values reveal this property. Can you reason out a solution to

$$\min_{\|x\|_2=1} \|Ax\|_2$$

in terms of the SVD? Can you construct a vector with unit 2-norm that gets mapped to a vector with this norm?