CS 3220: Prelim 1 example questions

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Whats in this document

These questions are intended to be somewhat representative of the type of questions that could arise on the first prelim.

Question 1

Here, we consider some properties of $\|A\|_F$. For what follows $A \in \mathbb{R}^{n\times n}$

(a) Prove that

$$\|A\|_F^2 = \sum_{i=1}^{n} \|A(:,i)\|_2^2.$$

(b) For any two orthogonal matrices $Q_1 \in \mathbb{R}^{n\times n}$ and $Q_2 \in \mathbb{R}^{n\times n}$ prove that

$$\|Q_1 A Q_2\|_F = \|A\|_F$$

(c) Prove that

$$\|A\|_F = \sqrt{\sum_{i=1}^{n} \sigma_i^2},$$

where $\sigma_1, \ldots, \sigma_n$ are the singular values of A.

Solution

(a) We have that $\|A\|_F^2 = \sum_{i,j} A_{i,j}^2$, which is equivalent to

$$\|A\|_F^2 = \sum_{j=1}^{n}\sum_{i=1}^{n} A_{i,j}^2.$$

The inner loop is $\|A(:,j)\|_2^2$, thereby completing the proof.

(b) Since the two norm of vectors is invariant under orthogonal transformations, the prior part gives us immediately that

$$\begin{aligned}
\|Q_1 A Q_2\|_F^2 &= \sum_{i=1}^{n} \|Q_1 (AQ_2)(:,i)\|_2^2 \\
&= \sum_{i=1}^{n} \|(AQ_2)(:,i)\|_2^2 \\
&= \|AQ_2\|_F^2.
\end{aligned}$$

Using the fact that for any matrix B, $\|B\|_F^2 = \|B^T\|_F^2$ we can use the same argument to remove Q_2.

(c) If $A = U\Sigma V^T$ we have that
$$\|A\|_F^2 = \|\Sigma\|_F^2.$$
Explicitly writing out the right hand side and taking the square root yields the result.

Question 2

Here, we ask you to interpret the condition number of a 2×2 matrix geometrically. (Hint: pictures are useful here!)

1. We saw that the SVD of a 2×2 matrix allows us to view any matrix A as mapping a circle to an ellipse. If A becomes increasing ill-conditioned what is geometrically happening to the ellipse?

2. Geometrically argue why for an ill-conditioned matrix a small relative change in b can result in a big relative change in x.

Solution

(a) The ratio of the length of the major axis to minor axis of the ellipse is going to infinity, so the ellipse is collapsing to a line segment.

(b) Because of the elongated shape of the ellipse, changes to b in the direction of v_2 can drastically alter the location along the ellipse.

Question 3

Say you are given a symmetric matrix A and tasked with computing the algebraically smallest eigenvalue. Using only the power method (applied to A or matrices related to A), how might you go about doing this? (Hint: think about how the eigenvalues/vectors of $A - \gamma I$ relate to those of A for any scalar $\gamma \in \mathbb{R}$.)

Solution

We can first use the power method to compute the largest eigenvalue in magnitude of A. If it is negative we are done, if not we need to do a bit more work. Let μ be the eigenvalue of A that we previously computed. Now, we can simply use the power method to compute an eigenvalue of $A - \mu I$, call it λ and then $\lambda + \mu$ is the algebraically smallest eigenvalue of A. This works because the eigenvalues of $A - \mu I$ are simply those of A shifted right by μ. Therefore the algebraically largest eigenvalue of A is 0 and the largest magnitude one is necessarily the smallest algebraic.