Question 1
Here, we consider some properties of $\|A\|_F$. For what follows $A \in \mathbb{R}^{n \times n}$

(a) Prove that
\[ \|A\|_F^2 = \sum_{i=1}^{n} \|A(:, i)\|_2^2. \]

(b) For any two orthogonal matrices $Q_1 \in \mathbb{R}^{n \times n}$ and $Q_2 \in \mathbb{R}^{n \times n}$ prove that
\[ \|Q_1 A Q_2\|_F = \|A\|_F \]

(c) Prove that
\[ \|A\|_F = \sqrt{\sum_{i=1}^{n} \sigma_i^2}, \]
where $\sigma_1, \ldots, \sigma_n$ are the singular values of $A$.

Solution
(a) We have that $\|A\|_F^2 = \sum_{i,j} A_{i,j}^2$, which is equivalent to
\[ \|A\|_F^2 = \sum_{j=1}^{n} \sum_{i=1}^{n} A_{i,j}^2. \]
The inner loop is $\|A(:, j)\|_2^2$, thereby completing the proof.

(b) Since the two norm of vectors is invariant under orthogonal transformations, the prior part gives us immediately that
\[ \|Q_1 A Q_2\|_F^2 = \sum_{i=1}^{n} \|Q_1(AQ_2)(:, i)\|_2^2 \]
\[ = \sum_{i=1}^{n} \|(AQ_2)(:, i)\|_2^2 \]
\[ = \|AQ_2\|_F^2. \]
Using the fact that for any matrix $B$, $\|B\|_F^2 = \|B^T\|_F^2$, we can use the same argument to remove $Q_2$. 

1
(c) If \( A = U\Sigma V^T \) we have that

\[ \|A\|_F^2 = \|\Sigma\|_F^2. \]

Explicitly writing out the right hand side and taking the square root yields the result.

**Question 2**

Here, we ask you to interpret the condition number of a \( 2 \times 2 \) matrix geometrically. (Hint: pictures are useful here!)

1. We saw that the SVD of a \( 2 \times 2 \) matrix allows us to view any matrix \( A \) as mapping a circle to an ellipse. If \( A \) becomes increasing ill-conditioned what is geometrically happening to the ellipse?

2. Geometrically argue why for an ill-conditioned matrix a small relative change in \( b \) can result in a big relative change in \( x \).

**Solution**

(a) The ratio of the length of the major axis to minor axis of the ellipse is going to infinity, so the ellipse is collapsing to a line segment.

(b) Because of the elongated shape of the ellipse, changes to \( b \) in the direction of \( v_2 \) can drastically alter the location along the ellipse.

**Question 3**

Say you are given a symmetric matrix \( A \) and tasked with computing the algebraically smallest eigenvalue. Using only the power method (applied to \( A \) or matrices related to \( A \)), how might you go about doing this? (Hint: think about how the eigenvalues/vectors of \( A - \gamma I \) relate to those of \( A \) for any scalar \( \gamma \in \mathbb{R} \).)

**Solution**

We can first use the power method to compute the largest eigenvalue in magnitude of \( A \). If it is negative we are done, if not we need to do a bit more work. Let \( \mu \) be the eigenvalue of \( A \) that we previously computed. Now, we can simply use the power method to compute an eigenvalue of \( A - \mu I \), call it \( \lambda \) and then \( \lambda + \mu \) is the algebraically smallest eigenvalue of \( A \). This works because the eigenvalues of \( A - \mu I \) are simply those of \( A \) shifted right by \( \mu \). Therefore the algebraically largest eigenvalue of \( A \) is 0 and the largest magnitude one is necessarily the smallest algebraic.