CS 3220: The condition number of a matrix

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Disclaimer

These notes are intended to highlight key points from some of the lectures. However, they are not written nor intended to be substitutes for the lectures. Furthermore, the resources section of the website contains extensive written material covering the topics in class, material that was developed with the explicit intention of being a written presentation of this material. Many of those books are several editions into their existence and, therefore, have been refined in a way these notes are not. In addition, the textbooks given are invariably far more comprehensive and thorough in their treatment of the topics. Lastly, I take full responsibility for any typos included herein; nevertheless, these notes are a work in progress and if you find anything amiss please let me know.

Conditioning of solving linear systems

In this brief note, we define the so-called condition number of a matrix and discuss its relevance to solving linear systems (a decidedly incomplete presentation of its value). However, this note actually proceeds in the reverse order and uses an analysis of the sensitivity of solutions to linear systems to motivate our ultimate definition of the condition number.

In linear algebra classes we learned that a $n \times n$ linear system of equations

$$Ax = b$$

with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ can have a unique solution, no solution, or an infinite number of solutions. Specifically, these scenarios were contingent on rather algebraically delicate properties of A and b. If A is full rank there is a unique solution, however if A is rank deficient (the aforementioned algebraically brittle condition) there is no solution if b is not in the range of A and an infinite number of solutions if b is in the range of A.

Here, we will instead be interested in asking the question

How sensitive is the solution to $Ax = b$ with respect to changes in b?

For simplicity we will hereafter assume that A is full rank (the implications of this will be clear later). The first part of this question we need to address is how we measure sensitivity. The metric we choose here is to measure the relative change in x normalized by the relative change in b, *i.e.,*

$$\frac{\|\widehat{x} - x\|_2/\|x\|_2}{\|\widehat{b} - b\|_2/\|b\|_2}$$

where $\widehat{x}$ is the solution to $A\widehat{x} = b$. The catch is that the above quantity depends on both the right hand side b and the specific perturbation $\widehat{b} - b$. To address the later we look at the worst case behavior over this quantity over all possible perturbations as

$$\max_{\widehat{b}} \frac{\|\widehat{x} - x\|_2/\|x\|_2}{\|\widehat{b} - b\|_2/\|b\|_2}.$$

Our goal now will be to bound the above in terms of properties of the matrix A—this will tell us something about the worst case behavior for fixed A and b.

To proceed we need to characterize the solutions x and $\widehat{x}$ in terms of b and $\widehat{b}$. While this class stresses that we solve linear systems (as opposed to inverting matrices), formally it will be useful to write $b = A^{-1}x$ since we are strictly considering the mathematical structure of the problem. Doing this yields

$$\begin{aligned}
\frac{\|\widehat{x}-x\|_2/\|x\|_2}{\|\widehat{b}-b\|_2/\|b\|_2} &= \frac{\|A^{-1}\widehat{b}-A^{-1}b\|_2/\|A^{-1}b\|_2}{\|\widehat{b}-b\|_2/\|b\|_2} \\
&= \frac{\|A^{-1}(\widehat{b}-b)\|_2/\|A^{-1}b\|_2}{\|\widehat{b}-b\|_2/\|b\|_2} \\
&\leq \frac{\|A^{-1}\|_2\|(\widehat{b}-b)\|_2\|b\|_2}{\|(\widehat{b}-b)\|_2\|A^{-1}b\|_2} \\
&\leq \|A^{-1}\|_2\frac{\|b\|_2}{\|A^{-1}b\|_2}.
\end{aligned}$$

The use of the sub-multiplicative bound $\|A^{-1}(\widehat{b}-b)\|_2 \leq \|A^{-1}\|_2\|(\widehat{b}-b)\|_2$ removes the dependence on $\widehat{b}$ and we can conclude that

$$\max_{\widehat{b}} \frac{\|\widehat{x}-x\|_2/\|x\|_2}{\|\widehat{b}-b\|_2/\|b\|_2} \leq \|A^{-1}\|_2\frac{\|b\|_2}{\|A^{-1}b\|_2}.$$

However, this still depends on the specific right hand side. To get a bound that depends only on properties of A we use that

$$\frac{\|b\|_2}{\|A^{-1}b\|_2} \leq \|A\|_2$$

to conclude that

$$\max_{\widehat{b},b} \frac{\|\widehat{x}-x\|_2/\|x\|_2}{\|\widehat{b}-b\|_2/\|b\|_2} \leq \|A\|_2\|A^{-1}\|_2.$$

The preceding arguments motivate the following definition (where we relate the norms of A and A^{-1} to singular values of A).

Definition 1 (Condition number)**.** *For any matrix $A \in \mathbb{R}^{n\times n}$ its condition number is defined as*

$$\kappa_2(A) = \frac{\sigma_1}{\sigma_n}.$$

By convention we say the condition number of A is infinite if A is singular (and hence $\sigma_n = 0$). This should seem reasonable given that if A is singular and a solution exists (*i.e.,* b is in the range of A) an infinite number exist. Therefore, one can change the solution without changing b at all. The key interpretation of this definition is that when solving a linear system with a large condition number the solution could be highly sensitive to perturbations/errors in b. Alternatively, when solving a system with a small condition number the solution is relatively insensitive to changes in b.