Problem 1: Toolbox (18 pts)

For all of the parts of this problem, you are limited to the following sets of tools:

(A) Backward Euler Method  (H) Midpoint Method  
(B) Bisection Method  (I) Newton’s Method  
(C) Condition Number  (J) QR Factorization  
(D) Divided differences  (K) Runge-Kutta 4/5 Method  
(E) Forward Euler Method  (L) Secant Method  
(F) Lagrange polynomials  (M) SVD  
(G) LU Factorization

For each of the following problem classifications, list all the tools from the above set that can be used to solve that type of problem. (Only include methods that are reasonably likely options—no need to look for tricky or obscure connections.)

a. Computing low-rank approximations to a data matrix
b. Solving initial-value ordinary differential equations
c. Finding least-squares solutions to systems of linear equations
d. Approximating functions with polynomials
e. Solving nonlinear scalar equations
f. Solving systems of nonlinear equations
g. Performing Principal Components Analysis
h. Solving systems of linear equations
Problem 2: MATLAB Code (16 pts)

Write efficient vectorized MATLAB code for each of the following expressions. Assume the cost of a matrix multiplication is independent of the contents of the matrices. All named matrices and vectors are stored in MATLAB arrays; the vectors are stored as column vectors.

a. \( y_i = A_{i,i}x_i \) for \( i = 1 \ldots n \), where \( A \) is an \( n \times n \) matrix.

b. \( B_{i,j} = \sum_{k=1}^{5} \sigma_k U_{i,k}V_{j,k} \) for \( i = 1 \ldots m \) and \( j = 1 \ldots n \), where \( U \) is an \( m \times n \) matrix, \( V \) is an \( n \times n \) matrix, and \( B \) is an \( m \times n \) matrix. Assume \( m, n \geq 5 \).

c. \( B_{i,j} = \sum_{k=1}^{n} M_{i,k}A_{k,j} \) for \( i = 1 \ldots n \) and \( j = 1 \ldots n \), where \( A \) and \( B \) are \( n \times n \) and \( M \) is \( n \times n \) and has the following structure:

\[
M = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & m_{4,3} & 1 & 0 & \cdots & 0 \\
0 & 0 & m_{5,3} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & m_{n,3} & 0 & 0 & \cdots & 1
\end{bmatrix}.
\]

d. \( c_i = \sum_{k=1}^{2n} A_{i,k}b_k \) for \( i = 1 \ldots 2n \), where \( b \) and \( c \) are \( 2n \)-vectors and \( A \) is a \( 2n \times 2n \) and block diagonal containing two copies of the \( k \times k \) matrix \( M \), in the form

\[
A = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}.
\]

The following (familiar, we hope) library functions may be of use: \( A = \text{ones}(m, n) \) to build a matrix of ones; \( A = \text{diag}(v) \) to build a diagonal matrix; \( v = \text{diag}(A) \) to extract the diagonal from a matrix; \( B = \text{reshape}(A, [m n]) \) to reformat the same data in a different-sized matrix. In these function signatures \( A \) and \( B \) are matrices, \( v \) is a vector, and \( m \) and \( n \) are integers.
Problem 3: Singular Value Decomposition (20 pts)

Consider the $2 \times 2$ matrices $A$, $B$, and $C$, which transform the unit circle into the following three shapes (the dot shows where a particular point is mapped to by each of the three transformations):

![Images of matrices A, B, and C transforming the unit circle]

a. The singular values of these matrices are all integers. What are the singular values of $A$, $B$, and $C$?

b. Write down the SVD of each of these matrices, choosing from the following matrices for $U$ and $V$:

(i) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
(ii) $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
(iii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(iv) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The matrix $M$ has the QR factorization

$$Q = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The matrix $N$ has the SVD

$$N = Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^T.$$

c. Which of the vectors $x = [0 \ 1 \ 1]^T$, $y = [1 \ 2 \ 2]^T$, and $z = [4 \ -1 \ -1]^T$, are elements of:

(i) $\text{ran}(M)$
(ii) $\text{ran}(N)$
(iii) $\text{null}(N)$
(iv) $\text{null}(M^T)$
**Problem 4: Stability and accuracy (15 pts)**

To learn about the error in our computed solution to an ordinary differential equation, we used three different methods to integrate the equation through a fixed time interval using a range of different step sizes. For each run we measured the error relative to a known analytic solution. The results are shown (schematically) in the following logarithmic plots:

![Graphs showing error in f(t) vs. step size for methods i, ii, and iii]

a. Order the three methods from most to least stable.

b. What is the order of accuracy of each method?

c. What method might each of these be, from among the ones we studied in class? (Just give one method for each.)
Problem 5: Convergence, and lack thereof (15 pts)

We used three different methods to numerically solve a particular nonlinear equation in one variable. We calculated the residual function value for increasing numbers of iterations and plotted it semilogarithmically:

![Graphs of residual function for methods i, ii, and iii]

a. Classify the convergence of each method as linear, sublinear, or superlinear.

b. Which methods are these, from among the ones we studied in class?

c. Give starting points for which each of the following root finding methods, when run on the function $f(x) = (x^2 + 1)/2$, will succeed for at least one iteration but blow up at a later iteration. Also explain what it does on the way to failing (i.e. what $x$s and $f(x)$s it encounters).

(i) Newton’s method (needs one starting point $x_0$)

(ii) Secant method (needs two starting points $x_0$ and $x_1$)
Problem 6: Condition numbers (16 pts)

Consider the matrix-vector multiplication:

\[
\begin{bmatrix}
3 & -3 & 5 \\
1 & 0 & -4 \\
5 & -3 & 3
\end{bmatrix}
\begin{bmatrix}
3 \\
5 \\
1
\end{bmatrix}
= \begin{bmatrix}
-1 \\
-1 \\
3
\end{bmatrix}
\]

\[A \times x = b\]

The condition number of \(A\) is 19.5, and its singular values are \(\sigma_1 = 9.28\), \(\sigma_2 = 4.07\), \(\sigma_3 = 0.476\).

a. If I change \(x\) to \(x + m\), where \(\|m\| < 0.01\), then do the matrix multiplication to update \(b\), what is the maximum relative change in \(b\) that could result?

b. If I change \(b\) to \(b + n\) where \(\|n\| < 0.01\), then solve the linear system to update \(x\), what is the maximum relative change in \(x\) that could result?

c. What bounds would the condition number give for both questions above?

In this problem, “relative change” means the norm of the change divided by the norm of the vector that is changing.