Linear Systems I (part 1)

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Outline

What is a linear system?

Examples of systems arising in applications

Systems of equations

A general category of numerical problem is *systems of equations* in which we have m statements about n variables $x_1, \dots, x_n$:

$$
\begin{aligned}
y_1 &= f_1(x_1, \dots, x_n) \\
y_2 &= f_2(x_1, \dots, x_n) \\
&\vdots \\
y_m &= f_m(x_1, \dots, x_n)
\end{aligned}
$$

The fs are known functions, the ys are given quantities, and the xs are the unknown variables to be determined.

Linear functions

A function f is linear if

$$f(x + x') = f(x) + f(x')$$

and

$$f(ax) = af(x)$$

for any a, x and x'.

If f is a function of one variable, it has to be $f(x) = ax$ for some a.

If f is a function of n variables it has the form

$$f(x1, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

Systems of linear equations

A linear system is just a system of equations

$$y_i = f_i(x_1, \ldots, x_n)$$

where the functions f_i that define it are linear. This means a system of m linear equations in n variables looks like this:

$$\begin{aligned}
y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
&\vdots \\
y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}$$

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$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

(I have switched to the convention for linear systems that the unknowns go on the right.)

Matrix form of linear system

A linear system can be written as one equation using matrix notation. A small example:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

becomes

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
$$

$$
\mathbf{A}\mathbf{x} = \mathbf{b}
$$

Matrix form of linear system

For those who enjoy dots:

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

A system of m equations in n unknowns is an equality with an m-vector on one side and the product of an $m \times n$ matrix with an n-vector on the other side.

Geometric transformations

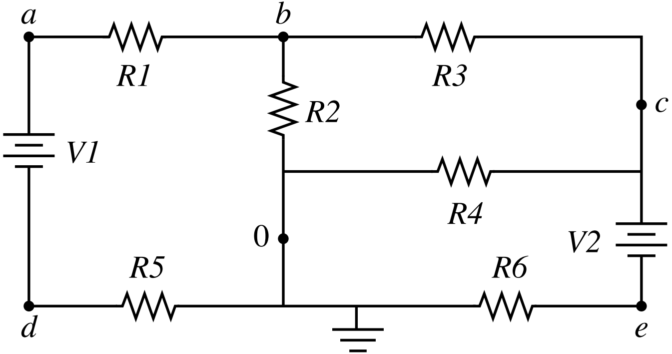
Suppose we have a coordinate system transformation:

$$x' = ax + by$$
$$y' = cx + dy$$

and we want to find the (x, y) that goes to (x', y'). This is a 2×2 linear system.

Circuit analysis

Here is a simple circuit:



What are the voltages across all the components? A straightforward analysis but tedious on paper.

Circuit analysis

- Variables
  - voltages at nodes: $v_a, \ldots, v_e$
  - currents through voltage sources I_1, I_2
- Equations
  - net current at each node is zero
  - known voltages across voltage sources

Result is a system with 7 variables and 7 equations

Radiative transfer

Problem in heat transfer: equilibrium energy distribution.

In, for example, a furnace, surfaces exchange heat by thermal radiation.

- Each surface emits radiation equally in all directions at some rate ϵ_i
- Each surface reflects a fraction ρ_i of the incident radiation
- The emitted radiation from one surface falls on the other surfaces
  - How it gets distributed depends on geometry
  - The fraction of light leaving patch i that ends up at patch j is the *form factor* f_{ij}.

Radiative transfer

A radiation balance at each surface results in a linear equation:

$$B_i = \epsilon_i + \rho_i \sum_j f_{ij} B_j$$

where B_i is the total radiation leaving the surface, known as its *radiosity* (hence the name of the method).