Problem set 1 question 1(25 points)

To solve for the coefficients in the Newton representation using InterpN.m, wee see that during pass k through the loop $2(n-k)$ subtractions and $(n-k)$ divisions are performed. So the total number of flops for this algorithm is

$$\begin{aligned}\sum_{k=1}^{n-1}(2(n-k)+(n-k)) &= \sum_{k=1}^{n-1}3n-3k = 3n(n-1)-3\sum_{k=1}^{n-1}k \\ = 3n^2-3n-3(n-1)(n)/2 &= \frac{3}{2}n^2-\frac{3}{2}n \approx \frac{3}{2}n^2\end{aligned}$$

HornerN evaluates the Newton representation polynomial with one subtraction, one multiplication, and one addition per degree of the polynomial. Hence the cost is $3(n-1)$ flops to evaluate the polynomial interpolant of n points. So the Newton method costs $\frac{3}{2}n^2$ flops (to leading term) to compute the coefficients and evaluate the polynomial once.

To evaluate the interpolant in the Lagrange representation, we see that for each of the n terms we have $2(n-1)$ subtractions (half in the numerator and half in the denominator), $2(n-2)+1$ multiplications ($(n-2)$ in the numerator, $(n-2)$ in the denominator and 1 to multiply by y_i), and one division. So the total number of flops is

$$n(2(n-1)+2(n-2)+1+1) = n(4n-4) = 4n^2-4n.$$

So solving for and evaluating the polynomial is less expensive in the Newton representation since Newton costs about $\frac{3}{2}n^2$ and Lagrange costs about $4n^2$. You could certainly save some flops in the Lagrange method, but it would still be an $O(n^2)$ calculation. To calculate the coefficients and evaluate the polynomial at n points would cost $\frac{3}{2}n^2+n\times(3n-3)\approx\frac{9}{2}n^2$ flops using the Newton representation, and $n\times(4n^2-4n)\approx 4n^3$ flops in the Lagrange representation. To evaluate at n^2 points would cost approximately $\frac{3}{2}n^2+n^2\times(3n-3)\approx 3n^3$ flops using the Newton representation, and $n^2\times(4n^2-4n)\approx 4n^4$ flops in the Lagrange representation. So although the Lagrange representation is elegant, it is less attractive (computationally) than the Newton representation.