Problem set 5
Due Friday, July 30 at 1:00 pm

Question 1 (30 points) Sparse QR factorization. If $A \in \mathbb{R}^{m \times n}$ is almost upper triangular (i.e. there are not very many non-zero sub-diagonal elements) Givens rotations can be selectively applied to zero the sub-diagonal entries. Suppose vectors $r$ and $p$ store the row and column of the non-zero sub-diagonal entries of $A$. In particular, entry $(r(i), p(i))$ of $A$ is nonzero for $i = 1 : k$ where $k$ is the length of $r$ and $p$. Write a matlab function SparseQR($A, r, p$) that returns the upper triangular matrix $R$ in the QR factorization of $A$ and vectors $c, s$ in $\mathbb{R}^k$ where $(c(i), s(i))$ are the $c$ and $s$ used in the Givens rotation zeroing $A(r(i), c(i))$. In this way the vectors $c$ and $s$ represent the orthogonal matrix $Q$ in the QR factorization. You may assume that input vector $p$ satisfies $p(1) \leq p(2) \leq \ldots \leq p(k)$. Efficiency matters. (Note that $A$ could be more efficiently stored, but you may assume it is stored as a full $m \times n$ matrix.) Hand in a print out of your matlab code and email a copy to gunsri@cs.cornell.edu

Question 2 (20 points) Consider the vectors $c$ and $s$ representing the Givens rotations applied to matrix $A$ in question 1. Write a matlab function SparseQMult($c, s, r, p, b$) that returns $Qb$, where $Q$ is the orthogonal matrix represented by $c, s$. Make sure to return $Qb$ and not $Q^Tb$. Efficiency matters. Hand in a print out of your matlab code and email a copy to gunsri@cs.cornell.edu

Question 3 (25 points) Let $a \in \mathbb{R}^n$ be considered as an $n \times 1$ matrix. Write out the QR factorization of $a$ showing the matrix $R$ explicitly and either give a formula for the matrix $Q$ or show each element explicitly (you can do this factorization in one step if you use a Householder transformation.) What is the solution to the linear least squares problem $ax \approx b$ where $b$ is a given $n$-vector? Show that the columns of $Q$ are an orthonormal basis for $\mathbb{R}^n$ and that $\frac{a}{\|a\|_2}$ is one of the basis vectors. This is an efficient way to build an orthonormal basis for $\mathbb{R}^n$ from an initial basis vector $a$.

Question 4 (10 points) If $A \in \mathbb{R}^{n \times n}$ is positive definite and $X \in \mathbb{R}^{n \times k}$ has rank $k$, prove that $X^TAX$ is positive definite.

Question 5 (15 points) If $A \in \mathbb{R}^{n \times n}$ is symmetric positive definite, prove that $|A(i, j)| \leq (A(i, i) + A(j, j))/2$. Hint: first establish the two inequalities $A(i, i) - 2A(i, j) + A(j, j) \geq 0$ and $A(i, i) + 2A(i, j) + A(j, j) \geq 0$. 

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