Problem set 4
Due Friday, July 23 at 1:00 pm

Question 1, part 1 (15 points) From your lecture notes, Gaussian elimination with partial pivoting in the presence of round-off error produces a solution $\hat{x}$ that satisfies $\frac{\|\hat{x}-x\|}{\|x\|} \approx$ $\texttt{eps} \cdot \kappa(A)$. To be more specific, usually $\frac{\|\hat{x}-x\|}{\|x\|}$ is about n times $\kappa(A)$, but in carefully contrived examples it can be as large as 2^n times $\kappa(A)$ (a big difference!) Consider matrices of the form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 1 \\ -1 & -1 & 1 & 0 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

Apply GEPP by hand to the 5×5 matrix above to compute the LU factorization. What do you observe?

Question 1, part 2 (25 points) Write a matlab script to apply Gaussian elimination with partial pivoting (you can use the function GEPiv from the book to do the GEPP) to various sizes ($n \in [5, 1000]$ at enough values for your plots to look nice) of linear systems of the form above, with right-hand-side vectors chosen so that you know the exact solution. Make plots (in matlab) showing how the error, residual, and condition number behave as the size of the system increases. Discuss briefly your observations. Hand in your plots and analysis with print-outs of your matlab scripts.

Question 2 (20 points) How would you efficiently solve a partitioned linear system of the form

$$\begin{bmatrix} L_1 & 0 \\ B & L_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$$

where L_1 and L_2 are nonsingular lower triangular matrices and the solution and right-hand-side vectors are partitioned accordingly? Show the specific steps you would perform in terms of the given submatrices and vectors. Compare the number of flops (leading term, including coefficient) that your method requires to solving the linear system (inefficiently) by applying Gaussian elimination to the matrix as a whole.

Question 3 (10 points) If A,B, and C are $n \times n$ matrices, with B and C nonsingular, and b is an n vector, how would you efficiently implement the formula

$$x = B^{-1}(2A + I)(C^{-1} + A)b$$

without computing any matrix inverses?

Question 4, part 1 (10 points) Suppose the matrix

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

is orthogonal, where the submatrices A and C are square. Prove that A and C must be orthogonal and $B = 0$.

Question 4, part 2 (10 points) Show that if the vector $v \neq 0$, then the matrix

$$H = I - 2\frac{vv^T}{v^T v}$$

is orthogonal and symmetric.

Question 4, part 3 (10 points) Let a be any nonzero vector. If $v = a - \alpha e_1$, where $\alpha = \pm\|a\|_2$, e_1 is the first column of the $n \times n$ identity matrix, and H is defined as in part 2, show that $Ha = \alpha e_1$.