Problem set 3

Due Monday, July 19 at 1:00 pm

Question 1 (25 points) Read section 4.4.1 in Van Loan about Gauss quadrature. Set up the equations for the weights w_1, w_2, w_3 and abscissas x_1, x_2, x_3 for the 3 point Gauss-Legendre rule (the rule that is exact for polynomials of degree 5 or less). These equations are nonlinear and difficult to solve. Use $x_1 = -.7745966692, x_2 = 0, x_3 = -x_1$ (the correct abscissas to the number of digits given) to simplify your nonlinear equations and solve for w_1, w_2, w_3. Use the Gauss-Legendre 3-point rule to approximate $\int_{-1}^{1} \frac{dx}{x+2}$. Compare your result to the approximation you get from Simpson's rule (which also requires three function evaluations) and to the exact value.

Question 2 (25 points) Recall the definition of an m-point quadrature rule (p. 136 in Van Loan). Prove that if Q is a given m-point quadrature rule (so the w_k and x_k are fixed) and Q_h represents the result of the quadrature rule applied to function $h(x)$ on $[a, b]$, then $Q_{f+g} = Q_f + Q_g$. That is, to calculate an approximation to $\int_a^b f(x) + g(x)dx$, you can add an approximation of $\int_a^b f(x)dx$ to an approximation of $\int_a^b g(x)dx$. Explain how you could efficiently calculate Q_{f-g} and Q_{f+g} if both values were desired. Assume, as usual, that the function evaluations are the dominant computational cost.

Question 3 (25 points) Write efficient matlab functions MatVecTriRO(l,d,u,x) and MatVecTriCO(l,d,u,x) to perform matrix-vector multiplications between a tridiagonal, square matrix A = diag(d) + diag(l,-1) + diag(u,1) and a vector x. Structure your computations in a row-oriented way in MatVecTriRO and in a column-oriented way in MatVecTriCO. Your functions must take advantage of the tridiagonal structure of A, must use vectorized operations, and must not form the matrix A explicitly. How many flops (highest order term, including coefficient) do your functions require for $A \in \Re^{n \times n}$, $x \in \Re^n$? How does this compare to calculating Ax by forming the matrix A (which requires no flops, but is a waste of memory) and then using MatVecRO or MatVecCO from chapter 5 of Van Loan? Hand in a printed copy of your functions and analysis and email your matlab functions to gunsri@cs.cornell.edu.

Question 4 (25 points) Prove the following relationships between the vector and matrix norms.

$$\begin{aligned} \|x\|_\infty \leq \|x\|_1 &\leq n\|x\|_\infty \\ \|x\|_\infty \leq \|x\|_2 &\leq \sqrt{n}\|x\|_\infty \\ \|Ax\|_2 &\leq \|A\|_2\|x\|_2 \end{aligned}$$

Hint: For the last relationship, recall that a norm on a vector space must satisfy $\|aZ\| = |a|\|Z\|$ for all scalars a and vectors Z. Also, note that in the last equation there are two different norms: on the LHS is the vector 2-norm, and on the RHS is the matrix 2-norm and the vector 2-norm.