Quiz #3
Due in class, Thursday, July 24, 2003

Guassian Quadrature

Answer all below questions. You may consult only the proctor. Submit all scratch paper and full, clear reasoning; this will help us give you partial credit. Let Π_n denote the collection of all polynomials of degree at most n.

1) Find x_1, x_2, if the integration formula

$$\int_{-1}^{1} f(x)dx \approx f(x_1) + f(x_2)$$

is to be exact for all $f \in \Pi_3$ and $x_2 = -x_1$.

2) *A more complete picture.* Orthogonal polynomials play a central role in the theory of Guassian quadrature. To every weight function, $\omega(x)$, $\omega(x) > 0$ on $[a, b]$, we associate an *inner product* $< \cdot, \cdot >_\omega$. We define this inner product by

$$< f, g >_\omega = \int_a^b f(x)g(x)\omega(x)dx.$$

Now we say that two functions f and g are ω-orthogonal if $< f, g >_\omega = 0$. Now we can construct (you will never need to) a sequence of polynomials $\{p_i\}_{i=0}^{\infty}$ with the property $< p_i, p_j >_\omega = 1$, if $j = i$ and $< p_i, p_j >_\omega = 0$, if $j \neq i$. We then say that this sequence is a sequence of ω-orthogonal polynomials. Every such sequence starts out with $p_0(x) = 1$ and next is $p_1(x) = x - c$, where c is some constant dependent upon our choice of $\omega(x)$.

While it had been evident that the weight function affects the "numbers" side of the nonlinear system (see example below), it was not apparent that the weight function had any effect on the "unknowns" side of the nonlinear system. The following discussion shows precisely how the weight function implicitly affects the "unknowns" side.

Example: The following is a cartoon of the nonlinear system generated by the 2-point Guassian rule with weight function ω.

$$\begin{array}{ccc} w_1 + w_2 & = & < 1, 1 >_\omega \\ \vdots & \vdots & \vdots \\ \text{"unknowns" side} & = & \text{"numbers" side} \\ \vdots & \vdots & \vdots \\ w_1 x_1^3 + w_2 x_2^3 & = & < x^3, 1 >_\omega = \int_a^b x^3 \omega(x)dx \end{array}$$

As was stated in class (and should be evident from the example), different choices of the weight function ω result in different numbers in the "numbers" side of the above system of nonlinear equations. As we have also seen in class the system greatly simplifies if we know, a priori, the x_i (recall the example from class) that gives the nice "exactness" property. It turns out that for an m-point Guassian rule, the roots of $p_m(x)$ give the x_i's that afford us this nice "exactness" property, where p_m is from the sequence of ω-orthogonal polynomials generated by the inner product $< \cdot, \cdot >_\omega$. No other choice of x_i's give us this property.

a) State the "exactness" property of an m-point Guassian quadrature rule.

b) Find x_i, w_i such that the approximation

$$\int_{-1}^{1} f(x)dx \approx w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

is exact for all $f \in \Pi_5$. The first ω-orthogonal polynomials are $p_0(x) = 1$, $p_1(x) = x$, $p_2(x) = x^2 - \frac{1}{3}$ and $p_3(x) = x^3 - \frac{3}{5}x$.