Midterm

Friday, July 25, 2003

Answer all below questions. You may consult only the proctor. Submit all scratch paper and full, clear reasoning; this will help us give you partial credit.

1) (34 pts) You have the three data points

$$(x_1,\, f(x_1)) = (0,\, 0),$$

$$(x_2,\, f(x_2)) = (\frac{\pi}{2},\, 1),$$

and

$$(x_3,\, f(x_3)) = (\pi,\, 0),$$

where f is the alleged underlying unknown function. Using the representation

$$p_{n-1}(x) = a_1 + a_2 x + \cdots + a_n x^{n-1},$$

set up, but do not solve, the equations that give the a_i's so that p_{n-1} interpolates f at the three data points. Design a 3-point interpolation function for this data based on the representation

$$q_{n-1}(x) = a_1 + a_2(\frac{x}{\pi}) + a_3(\frac{x}{\pi})^2 + \cdots + a_n(\frac{x}{\pi})^{n-1},$$

here $n \in \{1,\, 2,\, \ldots\}$. Be sure to explicitly write down your function. Explain your method's virtues, drawbacks. For this, using comparisons to other methods and quantified facts greatly bolsters your case.

Bonus: If possible, give interpretations of the unknowns in your method.

2) (33 pts) Clearly delineate how to find α as efficiently as possible given that

$$\alpha = b^T C^{-1} b,$$

where $C = AA^T$, matrices $A \in \mathcal{R}^{n\times n}$ and $C \in \mathcal{R}^{n\times n}$ are nonsingular and $b \in \mathcal{R}^n$. Assume that pivoting is not necessary. What does the cost of your method boil down to? Suppose we formed C, formed C^{-1}, then α, and we called this method 2. What would be the savings from your method, given above, over method 2 (Assume that forming C^{-1} from C requires $\frac{8}{3}n^3$ flops)? Note that forming C^{-1} from C is usually unstable and therefore a bad idea, but we consider it for sake of comparison. In general, when pivoting is necessary, why is it necessary?

3a) (23 pts) Find w_i, x_i, so that the integration formula

$$\int_{-1}^{1} x^2 f(x)dx = w_1 f(x_1) + w_2 f(x_2)$$

is exact for all $f \in \Pi_3$ (Hint: following the methodology in the text is one way to do it.).

3b) (10 pts) Let

$$I_1 = \frac{1}{3}\int_a^b (f - g - h)dx$$

and

$$I_2 = \frac{1}{3}\int_a^b (f + g + h)dx.$$

Clearly show how to estimate I_i ($i = 1, 2$) to within a tolerance of tol using Matlab's `quad` function and being as efficient as possible. Show why your approximations serve their purpose, clearly stating any properties you are using and why these properties are true.

Bonus. Discuss more reasons why your method wins over a brute-force method.