CS 322: Introduction to Scientific Computing
Spring 2003
Practice Final Exam

Handed out: Monday, April 28 (on the web).
This exam has 120 points. The students had 120 minutes to answer all questions. They were permitted to use an $8\frac{1}{2}'' \times 11''$ crib-sheet.

Note: one of the questions was a joke free giveaway question, so please do not send me email to ask about it. (You have to figure out which question was the joke giveaway.)

1. [**5 points**] What is meant when we say an $n \times n$ matrix A is nonsingular?

2. [**5 points**] What is does it mean for vectors to be "orthogonal"?

3. [**5 points**] Let $p(x)$ be a piecewise linear interpolant to the data $(x_1, y_1), \ldots, (x_n, y_n)$ where $x_1 < x_2 < \cdots < x_n$, so that $p(x_i) = y_i$ for $i = 1, \ldots, n$. Is there any circumstance in which $p(x)$ could have a continuous first derivative over the entire interval $[x_1, x_n]$?

4. [**5 points**] Let $p(x)$ be a degree-4 polynomial. Suppose we use an m-point Gauss quadrature rule to integrate p over $[0, 1]$. What is the minimum value of m for which this rule gives the exact answer?

5. [**5 points**] Write down an iterative method for computing the cube root of an arbitrary specified real number a. The method should use, on each iteration, simple operations like $+, -, \times, /$, and should use a as a coefficient. [Hint: Write down a nonlinear equation whose root is $a^{1/3}$, and then apply Newton's method to the equation.]

6. [**5 points**] Consider the IVP $y' = -y$, $y(0) = 1$. Note that the true solution (which is $y(t) = e^{-t}$) satisfies $y(t) > 0$ for all t. Show that the solution computed by EM also satisfies $y_n > 0$ for all n, provided that h is less than a certain constant. (What constant?) Assume h is fixed.

7. [**5 points**] Suppose a group of Numerical Analysts produces a major Hollywood movie called "Gauss and His Algorithm." Unfortunately the movie flops in all the cities where it opens: New York, London, Los Angeles, and Ithaca. How many flops is this (accurate to the leading term)?

8. [**10 points**] Describe the relative advantages and disadvantages of the method of normal equations versus QR factorization for solving least-squares problems.

9. [**15 points**] How many flops, accurate to the leading term, are required to compute the QR factorization via Givens rotations of an $n \times n$ matrix whose lower left $(n/2) \times (n/2)$ block is all zeros? In other words, the matrix has the following form:

$$A = \begin{pmatrix} U & V \\ 0 & W \end{pmatrix}$$

where U, V, W are each square $(n/2) \times (n/2)$ matrices.

10. **[10 points]** Write down the formula for $\mathbf{x}^{(k+1)}$ in terms of $\mathbf{x}^{(k)}$ that would be used for minimizing $f(x_1, x_2, x_3) = x_1^2 \cos x_2 + x_1 e^{x_3}$ using Newton's method.

11. **[15 points]** Consider solving the system of two nonlinear equations $f_1(x_1, x_2) = 0, f_2(x_1, x_2) = 0$. Note if (x_1^*, x_2^*) is a root to these equations, then it is also a root to the system $g_1(x_1, x_2) = 0, g_2(x_1, x_2) = 0$, where $g_1 = (f_1)^2$ and $g_2 = (f_2)^2$. But trying to find (x_1^*, x_2^*) by applying Newton's method (for rootfinding) to g would be a bad algorithm. Explain why. [Hint: Consider the Jacobian of g at the root.]

12. **[15 points]** Consider the following finite-difference method for integrating an IVP:

$$y_{n+1} = y_n + 0.5h(f(t_n, y_n) + f(t_{n-1}, y_{n-1})).$$

Show that this method is first-order, and in particular, find the local truncation error including its coefficient. (Assume h is fixed.)

13. **[10 points]** The method in the previous question is never used in practice because it has no major advantage, and yet it has a major disadvantage, when compared to Euler's method. Explain its relative advantages and disadvantages compared to Euler's method.

14. **[10 points]** Consider the IVP given by $y'(t) = c$, $y(0) = d$ (where c, d are constants). Show that for this IVP, Euler's method is exact (i.e., $y_n = y(t_n)$ for all n). Show that this is true regardless of stepsize.