Problem 1
Using the Uniform random variable $U$ on the interval $(0, 1)$, how would you express $W$, a random variable with the c.d.f. $F(w) = \begin{cases} 
0 & w \leq 0 \\
\frac{w^2}{2} & 0 \leq w \leq 1 \\
\frac{2 - (w - 2)^2}{2} & 1 \leq w \leq 2 \\
\frac{1}{2} & 2 < w
\end{cases}$
Can you express $W$ as a linear combination of $X$ and $Y$, 2 independant Uniform random variables each on the interval $(0, 1)$?

Solution 1
For $0 \leq w \leq 1$ we have $F(w) = \frac{w^2}{2}$ so in that range
$F^{-1}(F(w)) = \sqrt{2F(w)} = \sqrt{2\frac{w^2}{2}} = \sqrt{w^2} = w$, that is
$F^{-1}(F(w)) = \sqrt{2F(w)}$

When $0 \leq w \leq 1$, $0 \leq F(w) \leq \frac{1}{2}$ (plug in end points into c.d.f.)
For $1 \leq w \leq 2$ we have $F(w) = \frac{2 - (w - 2)^2}{2}$ so in that area
$F^{-1}(F(w)) = 2 - \sqrt{2 - 2F(w)} = 2 - \sqrt{2 - 2\frac{2 - (w - 2)^2}{2}} = 2 - \sqrt{2 - (2 - (w - 2)^2)} = 2 - \sqrt{(w - 2)^2} = 2 - (2 - w) = w.$
Notice that since $w \leq 2$, $(2 - w) \geq 0$ and hence $\sqrt{(w - 2)^2} = (2 - w)$.
So in this range
$F^{-1}(F(w)) = 2 - \sqrt{2 - 2F(w)}$

When $1 \leq w \leq 2$ as here, $F(1) = \frac{1}{2} \leq F(w) \leq 1 = F(2)$
By what we learnt in section, since $F$ is a c.d.f. and it has an inverse $F^{-1}$, then $W = F^{-1}(U)$ for a Uniform random variable $U$ on $(0, 1)$ has the distribution function $F$.

$W = F^{-1}(U) = \begin{cases} 
\sqrt{2U} & 0 \leq U \leq \frac{1}{2} \\
2 - \sqrt{2 - 2U} & \frac{1}{2} \leq U \leq 1
\end{cases}$

Notice that since $U$ is Uniform on $(0, 1)$ the range of our function is correct.

We have seen this exact c.d.f. $F(w)$ on homework1 for the random variable $W$ defined as $W = X + Y$ for $X$ and $Y$ being both Uniform on $(0, 1)$. 
Problem 2
Suppose that $X$ and $Y$ are independent random variables, each uniformly distributed on the interval $(0, 1)$. Let $W = X + Y$.
Calculate the mean of $W$ via linear combination of the means of $X$ and $Y$.
Calculate the variance of $W$ via linear combination of the variances of $X$ and $Y$.
Calculate the mean of $W$ directly by the definition (i.e. first calculate $f_W(w)$).
Calculate the variance $W$ directly by the definition (as above).

Solution 2
Let’s start by calculating $E(x)$ knowing that $f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$.

$$E(x) = \int_{-\infty}^{\infty} x * f_X(x) dx = \int_{0}^{1} x * 1 dx = \frac{x^2}{2}\bigg|_{0}^{1} = \frac{1}{2} - 0 = \frac{1}{2}$$

Now $E(W) = E(X + Y) = E(X) + E(Y) = \frac{1}{2} + \frac{1}{2} = 1$.

Let’s now calculate $Var(x)$. We must start with calculating $E(x^2)$.

$$E(x^2) = \int_{-\infty}^{\infty} x^2 * f_X(x) dx = \int_{0}^{1} x^2 * 1 dx = \frac{x^3}{3}\bigg|_{0}^{1} = \frac{1}{3} - 0 = \frac{1}{3}$$

So $Var(x) = E(x^2) - E^2(x) = \frac{1}{3} - (\frac{1}{2})^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Remember that $Var(x)$ is always positive!

Now $Var(W) = Var(X + Y) = Var(X) + Var(Y) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$.

From homework 1 we know the p.d.f. of $W$ to be $f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2 - w & 1 \leq w \leq 2 \\ 0 & otherwise \end{cases}$

So using the formula $E(w) = \int_{-\infty}^{\infty} w * f_W(w) dw$ we get

$$E(w) = \int_{0}^{1} w * wdw + \int_{1}^{2} w * (2 - w) dw = \int_{0}^{1} w^2 dw + \int_{1}^{2} 2w - w^2 dw =$$

$$\frac{w^3}{3}\bigg|_{0}^{1} + (w^2 - \frac{w^3}{3})\bigg|_{1}^{2} = (\frac{1}{3} - 0) + ((4 - \frac{8}{3}) - (1 - \frac{1}{3})) = \frac{1}{3} + \frac{2}{3} = 1$$

Which of course agrees with part a of this problem.

We will calculate $Var(w)$, by using the formula $Var(w) = E(w^2) - E^2(w)$

$$E(w^2) = \int_{0}^{1} w^2 * wdw + \int_{1}^{2} w^2 * (2 - w) dw = \int_{0}^{1} w^3 dw + \int_{1}^{2} 2w^2 - w^3 dw =$$

$$\frac{w^4}{4}\bigg|_{0}^{1} + (2w^3 - \frac{w^4}{4})\bigg|_{1}^{2} = (\frac{1}{4} - 0) + ((\frac{16}{3} - \frac{16}{4}) - (\frac{2}{3} - \frac{1}{4})) = \frac{3}{12} + \frac{8}{12} + \frac{3}{12} = \frac{14}{12} = \frac{7}{6}$$

Finally we get $Var(w) = E(w^2) - E^2(w) = \frac{7}{6} - 1^2 = \frac{1}{6}$, which as expected agrees with part b of this problem.
Problem 3

The formula for the p.d.f. of the distribution of a normal random variable $X$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Use the change of variable technique with $u = \frac{x-\mu}{\sqrt{2}\sigma}$ to find the mean of the normal distribution.

Assuming that the $X$ has mean $\mu$ and variance $\sigma^2$, how would you express the normal random variable $Z$ with mean 0 and variance 1 using linear operators on $X$?

Solution 3

If $u = \frac{x-\mu}{\sqrt{2}\sigma}$, then $du = \frac{dx}{\sqrt{2}\sigma}$ and $x = \mu + \sqrt{2}\sigma u$

We can now rewrite $E(x)$ using the new variable $u$. Notice that the limits of integration by $u$ and $x$ stay the same as they both grow in the range of $(-\infty, \infty)$.

$$E(x) = \int_{-\infty}^{\infty} x \left( \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx =$$

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma u) e^{-u^2} \sqrt{2} \sigma du = \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du + \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} ue^{-u^2} du$$

Now

$$\frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} ue^{-u^2} du = \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \left[ -\frac{1}{2} e^{-u^2} \right]_{-\infty}^{\infty} = \frac{-\sigma}{\sqrt{2\pi}} (0 - 0) = 0$$

Following the idea from class, we will look at $(\int_{-\infty}^{\infty} e^{-u^2} du)^2$ and then change the variables from Cartesian to polar coordinates (see side notes for slide 11):

$$\left( \int_{-\infty}^{\infty} e^{-u^2} du \right)^2 = \left( \int_{-\infty}^{\infty} e^{-u^2} du \right) \left( \int_{-\infty}^{\infty} e^{-y^2} dy \right) = \left( \int_{-\infty}^{\infty} e^{-x^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} dy \right)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} r dr d\theta = \int_{0}^{\infty} \int_{0}^{2\pi} 2\pi re^{-r^2} dr =$$

$$2\pi \int_{0}^{\infty} re^{-r^2} dr = 2\pi \left[ -\frac{1}{2} e^{-r^2} \right]_{0}^{\infty} = -\pi[0 - 1] = \pi$$

So $\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}$ and $\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} du = \frac{\mu}{\sqrt{\pi}} \sqrt{\pi} = \mu$

Summing it up we realize that

$$E(x) = \int_{-\infty}^{\infty} x \left( \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx = \mu$$

Similarly we can prove that

$$Var(x) = \int_{-\infty}^{\infty} x^2 \left( \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right) dx = \sigma^2$$
Using linearity of expectation we know that $X - E(x) = X - \mu$ will have expectation 0.

Using the rules $V(x - c) = V(x)$ and $V(ax) = a^2V(x)$ we can conclude that $Z = \frac{X - \mu}{\sigma}$ will have variance

$$V(Z) = V\left(\frac{X - \mu}{\sigma}\right) = V(X - \mu) \frac{1}{\sigma^2} = V(X) \frac{1}{\sigma^2} = \sigma^2 \frac{1}{\sigma^2} = 1$$

Notice that since the expectation of $X - \mu$ is zero then the division by $\sigma$ to $\frac{X - \mu}{\sigma}$ does not alter the expectation. So $Z = \frac{X - \mu}{\sigma}$ is distributed normally with mean 0 and variance 1.