Problem 1
Let \( f(x) = Ae^{-2x} \) for \( 0 < x < 2 \) (\( f(x) = 0 \) for any other value of \( x \)) be a p.d.f. For what value of \( A \) is \( f(x) \) a true density function? Using the value of \( A \) calculated above, what is the probability \( P(-2 \leq x \leq 2) \)?

Solution 1
For \( f \) to be a p.d.f. we must have 2 conditions:
\( f(x) \geq 0 \) for all \( x \) and \( \int_{\infty}^{-\infty} f(x)dx = 1. \)

\[
\int_{\infty}^{-\infty} f(x)dx = \int_{0}^{2} Ae^{-2x} dx = A(\left[-\frac{1}{2}e^{-2x}\right]_{0}^{2} = -\frac{1}{2}A(e^{-4} - e^{0}) = \frac{1}{2}A(1 - e^{-4})
\]

So since \( 1 = \int_{-\infty}^{\infty} f(x)dx = \frac{1}{2}A(1 - e^{-4}) \) we get that \( A = \frac{2}{(1-e^{-4})}. \)

We don’t really need the value of \( A \) calculated above to notice that \( f(x) > 0 \) only for \( 0 < x < 2 \) so \( P(0 \leq x \leq 2) = 1 \) and hence \( P(-2 \leq x \leq 2) = 1 \) too.
Problem 2

Suppose that $X$ is uniformly distributed on the interval $(-2, 3)$. Let $Y = X^2$.
Find the density function of $Y$.
Find the distribution function of $Y$.

Solution 2

Since $X$ is uniformly distributed, its p.d.f. should look like $f_X(x) = \begin{cases} A & x \in (-2, 3) \\ 0 & otherwise \end{cases}$

Since $1 = \int_{-\infty}^{\infty} f_X(x) \, dx = \int_{-2}^{3} A \, dx = Ax \big|_{-2}^{3} = A(3 - (-2)) = 5A$, we get that $A = \frac{1}{5}$. So the p.d.f. for $X$ is $f_X(x) = \begin{cases} \frac{1}{5} & x \in (-2, 3) \\ 0 & otherwise \end{cases}$

The c.d.f. of $X$ is simply an integration over the p.d.f.: $F_X(x) = \int_{-\infty}^{x} f_X(u) \, du$.

We must notice that this function has 3 distinct areas: $x < -2$, $x > 3$ and $-2 \leq x \leq 3$. For $x < -2$, $F_X(x) = 0$ clearly. For $x > 3$, $F_X(x) = 1$. The interesting part is for $-2 \leq x \leq 3$ where we get:

$$F_X(x) = \int_{-\infty}^{x} f_X(u) \, du = \int_{-2}^{x} \frac{1}{5} \, du = \frac{1}{5} x^{\lfloor x \rfloor} = \frac{1}{5} (x - (-2)) = \frac{1}{5} (x + 2)$$

So our c.d.f. is $F_X(x) = \begin{cases} 0 & x < -2 \\ \frac{1}{5} (x + 2) & -2 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$

Now $Y = X^2$, so if $x \in (-2, 3)$ then $y \in (0, 9)$. We will split the problem into 2 areas: $0 \leq y \leq 4$ (i.e. where $-2 \leq x \leq 2$) and $4 < y \leq 9$ (i.e. for $2 < x \leq 3$).

Let's calculate the c.d.f. for $y \in (0, 4)$.

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = P(X \leq \sqrt{y}) - P(X \leq -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

Since $y \in (0, 4)$ we get that $x = \sqrt{y} \in (0, 2)$ and $x = -\sqrt{y} \in (-2, 0)$ so $x \in (-2, 2)$ and we can plug it into the c.d.f. from above to get

$$F_X(\sqrt{y}) - F_X(-\sqrt{y}) = \frac{1}{5} (\sqrt{y} + 2) - \frac{1}{5} (-\sqrt{y} + 2) = \frac{2\sqrt{y}}{5}$$

For $y \in (4, 9)$ the situation is simpler, since we definitely know that $x \in (2, 3)$.

$$F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = P(X \leq \sqrt{y}) = F_X(\sqrt{y}) = \frac{1}{5} (\sqrt{y} + 2)$$

We combine our results and get $Y$'s c.d.f. to be: $F_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{2\sqrt{y}}{5} & 0 \leq y \leq 4 \\ \frac{1}{5} (\sqrt{y} + 2) & 4 < y \leq 9 \\ 1 & y > 9 \end{cases}$

For the p.d.f. we can derive the c.d.f. and get $f_Y(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{10\sqrt{y}} & 0 \leq y \leq 4 \\ \frac{1}{10\sqrt{y}} & 4 < y \leq 9 \\ 0 & otherwise \end{cases}$
Problem 3
Suppose that $X$ and $Y$ are independent random variables, each uniformly distributed on the interval $(0, 1)$. Let $V = X - Y$ and let $W = X + Y$.
Find the distribution function of $V$.
Find the distribution function of $W$.

Solution 3
We have already seen that the p.d.f. and c.d.f. of a uniformly distributed random variables such as $X$ and $Y$ should be:

$$f(x) = \begin{cases} 
1 & x \in (0, 1) \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad F(x) = \begin{cases} 
0 & x < 0 \\
x & x \in (0, 1) \\
1 & x > 1
\end{cases}$$

For any value $v$ of $V = X - Y$ there are many options of what values $X$ and $Y$ can get. We can be more precise and determine that for any value $v$, if $X = x$ then $Y$ must equal $(x - v)$ so that $V = X - Y = x - (x - v) = v$.
To get the p.d.f. of $V$ at point $v$ we will then integrate over all possible values of $x$. The p.d.f. can be written as $f_V(v) = \int_{-\infty}^{\infty} f_X(x) f_Y(x-v) \, dx$
Let's look at the p.d.f. of $V$ when $v$ is positive ($0 \leq v \leq 1$). Notice that while both $x$ and $y$ are in $(0, 1)$, $v$ can get values between -1 and 1.
Our multiplicants are non zero when $0 \leq x \leq 1$ and $0 \leq x - v \leq 1$ which in this case ($0 \leq v \leq 1$) means $v \leq x \leq 1 + v$. So the integration limits will be $v \leq x \leq 1$:

$$f_V(v) = \int_v^1 f_X(x) f_Y(x-v) \, dx = \left[ x \right]_v^1 1 \, dx = \int_v^1 1 \, dx = x|_v^1 = 1 - v$$

When $-1 \leq v \leq 0$, our integration limits change into $0 \leq x \leq 1 - v$ and we get:

$$f_V(v) = \int_0^{1-v} f_X(x) f_Y(x-v) \, dx = \left[ x \right]_0^{1-v} 1 \, dx = \int_0^{1-v} 1 \, dx = x|_0^{1-v} = v + 1$$

We combine our results and get the p.d.f. $f_V(v) = \begin{cases} 
v + 1 & -1 \leq v \leq 0 \\
1 - v & 0 \leq v \leq 1 \\
0 & \text{otherwise}
\end{cases}$

To get the c.d.f. we must integrate the p.d.f. to get:
For $v \in (-1, 0)$,

$$F_V(v) = \int_{-1}^{v} f_V(u) \, du = \int_{-1}^{v} (u+1) \, du = \left[ \frac{u^2}{2} + u \right]_{-1}^{v} = \frac{v^2}{2} + v - \frac{1}{2} - 1 = \frac{v^2}{2} + v + \frac{1}{2}$$

For $v \in (0, 1)$,

$$F_V(v) = \int_{-1}^{0} f_V(u) \, du + \int_{0}^{v} (1-u) \, du = \left[ \frac{1}{2} + \frac{u^2}{2} \right]_{0}^{v} = \frac{1}{2} + v - \frac{v^2}{2}$$

And finally $F_V(v) = \begin{cases} 
0 & v < -1 \\
\frac{v^2}{2} + v + \frac{1}{2} & -1 \leq v \leq 0 \\
\frac{v^2}{2} + v - \frac{v^2}{2} & 0 \leq v \leq 1 \\
1 & 1 < v
\end{cases}$
\( W = X + Y \) is very similar to calculate.

For any value \( w \) of \( W = X + Y \) there are many options of what values \( X \) and \( Y \) can get. For any value \( w \), if \( X = x \) then \( Y \) must equal \((w - x)\) so that \( W = X + Y = x + (w - x) = w \).

To get the p.d.f. of \( W \) at point \( w \) we will then integrate over all possible values of \( x \). The p.d.f. can be written as \( f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) \, dx \)

Notice that this time since both \( x \) and \( y \) are in \((0, 1)\), \( w \) can get values between 0 and 2.

Our multiplicants are non zero when \( 0 \leq x \leq 1 \) and \( 0 \leq w - x \leq 1 \) which in our case means \( w - 1 \leq x \leq w \).

Again it will be comfortable splitting the function into 2 areas namely \( w \in (0, 1) \) and \( w \in (1, 2) \) so that our integration includes non-zero values only.

For \( w \in (0, 1) \) the integration limits will be \( 0 \leq x \leq w \):

\[
 f_W(w) = \int_{0}^{w} f_X(x) f_Y(w - x) \, dx = \int_{0}^{w} 1 \, dx = x \bigg|_{0}^{w} = w
\]

For \( w \in (1, 2) \) the integration limits will be \( w - 1 \leq x \leq 1 \) and we get:

\[
 f_W(w) = \int_{w-1}^{1} f_X(x) f_Y(w - x) \, dx = \int_{w-1}^{1} 1 \, dx = x \bigg|_{w-1}^{1} = 2 - w
\]

We combine our results and get the p.d.f. \( f_W(w) = \begin{cases} w & 0 \leq w \leq 1 \\ 2 - w & 1 \leq w \leq 2 \\ 0 & \text{otherwise} \end{cases} \)

To get the c.d.f. we must integrate the p.d.f. to get:

For \( w \in (0, 1) \),

\[
 F_W(w) = \int_{0}^{w} f_W(u) \, du = \int_{0}^{w} u \, du = \frac{u^2}{2} \bigg|_{0}^{w} = \frac{w^2}{2} - 0 = \frac{w^2}{2}
\]

For \( w \in (1, 2) \),

\[
 F_W(w) = \int_{0}^{1} f_W(u) \, du + \int_{1}^{w} f_W(u) \, du = \frac{1}{2} + \int_{1}^{w} (2 - u) \, du = \frac{1}{2} + \left( 2u - \frac{u^2}{2} \right) \bigg|_{1}^{w} = \frac{1}{2} + (2w - \frac{w^2}{2}) - (2 - \frac{1}{2}) = 2w - \frac{w^2}{2} - 1
\]

Finally we get \( F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{w^2}{2} & 0 \leq w \leq 1 \\ 2w - \frac{w^2}{2} - 1 & 1 \leq w \leq 2 \\ 1 & 2 < w \end{cases} \)

Note: There are many other ways of finding these functions.
Problem 4
Suppose that $X$ and $Y$ are independent random variables and each is of exponential distribution with mean $\frac{1}{3}$, i.e. $f(x) = 3e^{-3x}$ and $f(y) = 3e^{-3y}$. Let $V = X + Y$, let $W = \min(X,Y)$ and let $Z = \max(X,Y)$.

Find the distribution function of $V$.
Find the distribution function of $W$.
Find the distribution function of $Z$.

Solution 4
For $V = X + Y$ we will follow the same principle as before:
For any value $v$ of $V = X + Y$ there are many options of what values $X$ and $Y$ can get. For any value $v$, if $X = x$ then $Y$ must equal $(v - x)$ so that $W = X + Y = x + (v - x) = w$.

To get the p.d.f. of $V$ at point $v$ we will then integrate over all possible values of $x$. The p.d.f. can be written as $f_V(v) = \int_{-\infty}^{\infty} f_X(x) f_Y(v - x)dx$
Notice that this time since both $x$ and $y$ are in $(0,\infty)$, $v = x + y$ also has the same range.

Our multiplicants are non zero when $0 \leq x \leq \infty$ and $0 \leq v - x \leq \infty$ which in our case means $-\infty \leq x \leq v$. Combining these 2 conditions we get that the integration limits should be $0 \leq x \leq v$:

$$f_V(v) = \int_{0}^{v} f_X(x) f_Y(v - x)dx = \int_{0}^{v} 3e^{-3x} * 3e^{-3(v - x)} dx = \int_{0}^{v} 9e^{-3v} dx = 9e^{-3v}$$

To get the c.d.f. we must integrate the p.d.f. to get:

$$F_V(v) = \int_{0}^{v} f_V(u)du = \int_{0}^{v} 9ue^{-3u} du =$$

To evaluate $\int_{0}^{v} 9ue^{-3u} du$ we will use integration by parts ($\int xdy = xy - \int ydx$).
Set $x = 9u$ and $dy = e^{-3u} du$. So $dx = 9du$ and $y = \int e^{-3u} du = \frac{-1}{3} e^{-3u}$
Now integrate

$$\int_{0}^{v} 9ue^{-3u} du = (9u * \frac{-1}{3} e^{-3u})_{0}^{v} - \int_{0}^{v} \frac{-1}{3} e^{-3u} * 9du = -3ue^{-3u}|_{0}^{v} - \int_{0}^{v} 3e^{-3v} dv$$

$$= (-3ve^{-3v} - 0) - (-e^{-3u})_{0}^{v} = -3ve^{-3v} - (e^{-3v} - (-1)) = 1 - (3v + 1)e^{-3v}$$

So $F_V(v) = \begin{cases} 
0 & v < 0 \\
1 - (3v + 1)e^{-3v} & v > 0
\end{cases}$

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Let’s start with $Z = \max(X, Y)$. What does the fact that $Z$ is $\max(X, Y)$ mean?

It means that if $Z \leq z$ for some $z$, then both $X$ and $Y$ are smaller than $z$.

In probability notation we can say that $P(Z \leq z) = P(X \leq z \text{ and } Y \leq z)$. Since $X$ and $Y$ are independent we can write this as $P(Z \leq z) = P(X \leq z) P(Y \leq z)$.

By definition $P(X \leq z) = F_X(z)$ and $P(Y \leq z) = F_Y(z)$.

So all we need to do is find the c.d.f. of the exponential function.

$$F(x) = \int_0^x 3e^{-3u} = -e^{-3u} \bigg|_0^x = (-e^{-3v} - 1) = 1 - e^{-3v}$$

So the c.d.f. is (for $z \geq 0$):

$$F_Z(z) = P(Z \leq z) = P(X \leq z) \cdot P(Y \leq z) = F_X(z) \cdot F_Y(z) = (1 - e^{-3z})^2 = 1 - 2e^{-3z} + e^{-6z}$$

With $W = \min(X, Y)$ we go through a similar process.

Here, when $W \leq w$, it means that the minimum of $X$ and $Y$ is less than $w$. We can rephrase it as when the minimum is greater than $w$ ($W > w$), then both $X$ and $Y$ must be also greater than $w$. To put this in probabilistic notation: $P(W > w) = P(X > w \text{ and } Y > w) = P(X > w) P(Y > w)$ since $X$ and $Y$ are independent.

We know that $P(W \leq w) = F_W(w)$, but what about $P(W > w)$?

$P(W > w)$ is the complement of $P(W \leq w)$, so $P(W > w) = 1 - P(W \leq w)$

Now we are ready to calculate our c.d.f.

$$F_W(w) = P(W \leq w) = 1 - P(w > w) = 1 - P(X > w) P(Y > w)$$

$P(X > w) = 1 - P(X \leq w) = 1 - F_X(w)$ for the same reasons as before and we finally get:

$$F_W(w) = 1 - P(X > w) P(Y > w) = 1 - (1 - P(X \leq w))(1 - P(Y \leq w)) =$$

$$1 - (1 - F_X(w)(1 - F_Y(w)) = 1 - (1 - F_X(w) - F_Y(w) + F_X(w)F_Y(w) =$$

$$1 - (1 - (1 - e^{-3w}) - (1 - e^{-3w}) + (1 - e^{-3w})^2) =$$

$$1 - (1 - 1 + e^{-3w} - 1 + e^{-3w} + 1 - 2e^{-3w} + e^{-6w}) = 1 - e^{-6w}$$

Here too, the domain is $(0, \infty)$.

Note: There are many other ways of finding these functions.