As discussed in section, a minimization algorithm cannot solve the constrained minimization problem in a straightforward way. In other words, if we wish to minimize a function $f$ subject to some constraint $\sigma = 0$, then the method of Lagrange multipliers says that the solution will be a stationary point of $g = f + \lambda \sigma$. To find the stationary point we minimize the norm of the gradient of $g$, $\| \nabla g \|^2$. We will do this for $f(x, y) = x + y$ and $\sigma(x, y) = x^2 + y^2 - 1$.

1. Solve the problem without doing any calculations. That is, write down the point $(x_{\min}, y_{\min})$ that minimizes $f$ (subject to $\sigma = 0$) using pictures.
   **Answer:** The level lines of $f$ are of the form $y = -x + c$. The smaller $c$ is, the lower the line is in the $xy$-plane. The minimum is therefore the lower-leftmost point on the circle $\sigma$: $(x_{\min}, y_{\min}) = (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.

2. Solve for $(x_{\min}, y_{\min})$ analytically and verify your answer in part 1.
   **Answer:** The equations
   \[
   \frac{\partial g}{\partial x} = 1 + 2\lambda x = 0 \\
   \frac{\partial g}{\partial y} = 1 + 2\lambda y = 0 \\
   \frac{\partial^2 g}{\partial x^2} = x^2 + y^2 - 1 = 0
   \]
   give the two solutions $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. The first is clearly a maximum of $f$ on the circle, and the second is the minimum $(x_{\min}, y_{\min})$. We also find $\lambda = \frac{1}{\sqrt{2}}$.

3. Evaluate the Hessian of $g$ at $(x_{\min}, y_{\min})$ and show that it is not positive definite.
   **Answer:** The Hessian is given by
   \[
   \begin{pmatrix}
   \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial y \partial x} & \frac{\partial^2 g}{\partial \lambda \partial x} \\
   \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} & \frac{\partial^2 g}{\partial \lambda \partial y} \\
   \frac{\partial^2 g}{\partial x \partial \lambda} & \frac{\partial^2 g}{\partial y \partial \lambda} & \frac{\partial^2 g}{\partial \lambda^2}
   \end{pmatrix}
   = 2 \begin{pmatrix}
   \lambda & 0 & x \\
   0 & \lambda & y \\
   x & y & 0
   \end{pmatrix}
   \]
   The eigenvalues $e$ of this matrix (ignoring the factor of 2) are given by the cubic equation
   \[
   (\lambda - e)[-e(\lambda - e) - y^2 - x^2] = 0
   \]
   Clearly, one of the eigenvalues is the Lagrange multiplier $\lambda$. The other two can be found by solving the quadratic equation
   \[
   -e(\lambda - e) - y^2 - x^2 = 0
   \]
   which reduces to $e^2 - e \lambda - 1 = 0$ after we use the constraint $x^2 + y^2 = 1$. The
quadratic formula then gives us
\[ e = \frac{\lambda \pm (\lambda^2 + 4)^{1/2}}{2} \]
which shows that one of the eigenvalues is negative (recall \( \lambda = \frac{1}{\sqrt{2}} \)). The Hessian is therefore not positive definite, and the point \((x_{\min}, y_{\min})\) is a saddle point of \(g\), not an extremum.

4. Now use the Matlab function \texttt{fminunc} to compute \((x_{\min}, y_{\min})\) by minimizing \(|\nabla g|^2\).

\textbf{Answer:} Here is the matlab function \texttt{hw4func}, which evaluates \(|\nabla g|^2\):

```matlab
function gg2 = hw4func(p)

lambda = p(3);
y = p(2);
x = p(1);

sigma = x^2 + y^2 - 1;
f = x+y;
g = f + lambda * sigma;
gradf = [1+2*lambda*x 1+2*lambda*y];
grdg = [gradf sigma];
gg2 = gradg * gradg';

And the call to \texttt{fminunc} (with starting point \(x=-1, y=-1, \lambda = 1\)) produces the following:

\[
\texttt{\textgreater\textgreater x = fminunc('hw4func', [-1 -1 1])}
\]
\[
\texttt{[[miscellaneous junk]]}
\]
\[
\texttt{x = -0.7071 -0.7071 0.7071}
\]