CS 3110

Induction in Coq

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Review

Previously in 3110:

- Functional programming in Coq
- Logic in Coq
- Curry-Howard correspondence (proofs are programs)

Today:

- Induction in Coq

REVIEW:
INDUCTION ON NATURAL NUMBERS AND LISTS

Structure of inductive proof

Theorem:
for all natural numbers n, P(n).

Proof: by induction on n

Case: n = 0
Show: P(0)

Case: n = k+1
IH: P(k)
Show: P(k+1)

QED

Sum to n

```
let rec sum_to n =
  if n=0 then 0
  else n + sum_to (n-1)
```

$$\sum_{i=0}^{n} i$$

Theorem:

for all natural numbers n,
  sum_to n = n * (n+1) / 2.

Proof: by induction on n

P(n) ≡ (sum_to n = n * (n+1) / 2)

```
let rec sum_to n =
  if n=0 then 0
  else n + sum_to (n-1)
```

Base case

Case: `n = 0`

Show:

```
   P(0)
≡  sum_to 0 = 0 * (0+1) / 2
≡  0        = 0 * (0+1) / 2
≡  0        = 0
```

```
let rec sum_to n =
  if n=0 then 0
  else n + sum_to (n-1)
```

Inductive case

Case: n = k+1

IH: P(k) ≡ sum_to k = k * (k+1) / 2

Show:

```
  P(k+1)
≡ sum_to (k+1)             = (k+1) * (k+2) / 2
≡ (k+1) + sum_to (k+1-1)   = (k+1) * (k+2) / 2
≡ (k+1) + sum_to k         = (k+1) * (k+2) / 2
≡ (k+1) + k * (k+1) / 2    = (k+1) * (k+2) / 2
```

and that holds by algebraic reasoning

QED

Structure of inductive proof

Theorem:
for all natural numbers n, P(n).

Proof: by induction on n

Case: n = 0
Show: P(0)

Case: n = k+1
IH: P(k)
Show: P(k+1)

QED

Structure of inductive proof

Theorem:
for all lists lst, P(lst).

Proof: by induction on lst

Case: lst = []
Show: P([])

Case: lst = h::t
IH: P(t)
Show: P(h::t)

QED

Append nil

```
let rec (@) lst1 lst2 =
  match lst1 with
  | []   -> lst2
  | h::t -> h :: (t @ lst2)
```

Theorem:
for all lists lst, lst @ [] = lst.

Proof: by induction on lst

P(lst) ≡ lst @ [] = lst

```
let rec (@) lst1 lst2 =
  match lst1 with
  | []   -> lst2
  | h::t -> h :: (t @ lst2)
```

Base case

Case: `lst = []`

Show:

`P([])`

≡ *`[] @ [] = []`*

≡ *`[]`* `= []`

```
let rec (@) lst1 lst2 =
  match lst1 with
   | []   -> lst2
   | h::t -> h :: (t @ lst2)
```

Inductive case

```
P(lst) ≡ lst @ [] = lst
```

Case: `lst = h::t`

IH: `P(t) ≡ t @ [] = t`

Show:

```
  P(h::t)
≡ (h::t) @ [] = h::t
≡ h::(t @ []) = h::t
≡ h::t        = h::t
```

QED

Append nil in Coq

```
Theorem app_nil :
  forall (A:Type) (lst : list A),
  lst ++ nil = lst.
Proof.
  intros A lst.
  induction lst as [ | h t IH].
  - trivial.
  - simpl. rewrite -> IH. trivial.
Qed.
```

Append nil in Coq

```
Theorem app_nil :
  forall (A:Type) (lst : l
  lst ++ nil = lst.
Proof.
  intros A lst.
  induction lst as [ | h t IH].
  - trivial.
  - simpl. rewrite -> IH. trivial.
Qed.
```

++ is append operator in Coq

base case: nothing to name

inductive case: name head, tail, and inductive hypothesis

`rewrite ->` tactic replaces LHS of equality with RHS

Append is associative

```
Theorem app_assoc :
  forall (A:Type) (l1 l2 l3 : list A),
  l1 ++ (l2 ++ l3) = (l1 ++ l2) ++ l3.
Proof.
  intros A l1 l2 l3.
  induction l1 as [ | h t IH].
  - trivial.
  - simpl. rewrite -> IH. trivial.
Qed.
```

INDUCTION ON NATS

Inductive types

`induction` works on inductive types, e.g.

```
Inductive list (A : Type) : Type :=
  | nil : list A
  | cons : A -> list A -> list A
```

Need an inductive definition of natural numbers…

Naturals

```
Inductive nat : Set :=
  | O : nat                (* zero *)
  | S : nat -> nat         (* succ *)
```

```
type nat = O | S of nat
```

```
0 is O
1 is S O
2 is S (S O)
3 is S (S (S O))
```

- unary representation
- Peano arithmetic

Induction on nat(ural)s

Theorem:
for all n:nat, P(n)
Proof: by induction on n

Case: n = O
Show: P(O)

Case: n = S k
IH: P(k)
Show: P(S k)

QED

Theorem:
for all naturals n, P(n)
Proof: by induction on n

Case: n = 0
Show: P(0)

Case: n = k+1
IH: P(k)
Show: P(k+1)

QED

Goal: redo this proof in Coq

```
let rec sum_to n =
  if n=0 then 0
  else n + sum_to (n-1)
```

$$\sum_{i=0}^{n} i$$

Theorem:

for all natural numbers n,
  sum_to n = n * (n+1) / 2.

Proof: by induction on n

Defining sum_to

```
Fixpoint sum_to (n:nat) : nat :=
  if n = 0 then 0
  else n + sum_to (n-1).
Error: The term "n = 0" has type "Prop" which
is not a (co-)inductive type.
```

```
Fixpoint sum_to (n:nat) : nat :=
  if n =? 0 then 0
  else n + sum_to (n-1).
Recursive definition of sum_to is ill-formed.
...
Recursive call to sum_to has principal argument
 equal to "n - 1" instead of a subterm of "n".
```

No infinite loops

```
Fixpoint inf (x:nat) : nat :=
  inf x.
Recursive definition of inf is ill-formed.
...
Recursive call to inf has principal argument
 equal to "x" instead of a subterm of "x".
```

Why no infinite loops?

In OCaml:

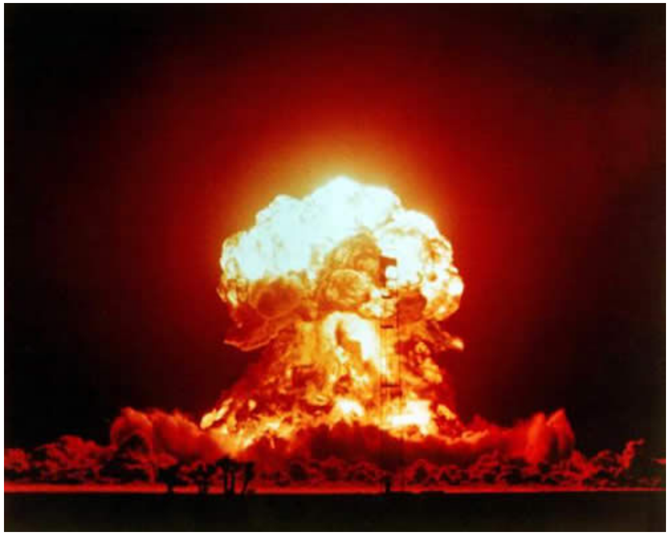
```
# let rec inf x = inf x
val inf : 'a -> 'b = <fun>
```

By propositions-as-types, these are the same:

- `'a -> 'b`
- $A \Rightarrow B$

What if A=True, B=False?

Infinite loops prove False!



Defining sum_to

```
Fixpoint sum_to (n:nat) : nat :=
  match n with
  | 0   => 0
  | S k => n + sum_to k
  end.
```

```
sum_to is defined
```

k is a subterm of n,
because n = S k,

Sum to n in Coq

```
Theorem sum_sq_no_div :
  forall n : nat,
  2 * sum_to n = n * (n+1).
Proof.
  intros n.
  induction n as [ | k IH].
  - trivial.
  - rewrite -> sum_helper.
    rewrite -> IH.
    ring.
Qed.
```

base case: nothing to name

inductive case: name inner nat and inductive hypothesis

tactic that finds proofs for algebraic equations on rings

Helper theorem

Lemma and Theorem are synonymous

```
Lemma sum_helper :
  forall n : nat,
  2 * sum_to (S n) = 2 * S n + 2 * sum_to n.
Proof.
  intros n. simpl. ring.
Qed.
```

Induction and recursion

- Intense similarity between inductive proofs and recursive functions on variants
  - In proofs: one case per constructor
  - In functions: one pattern-matching branch per constructor
  - In proofs: uses IH on "smaller" value
  - In functions: uses recursive call on "smaller" value

- Proofs = programs
- Inductive proofs = recursive programs

Upcoming events

- [next Tuesday] Prelim II
- [next Wednesday] A5 out
- [next Friday] Yaron Minsky talk