CS 3110

The Substitution Model

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Review

Previously in 3110: simple interpreter for expression language

- abstract syntax tree (AST)
- evaluation based on single steps
- parser and lexer (in lab)

Today:

- Formal syntax: BNF
- Formal dynamic semantics: small-step, substitution model
- Formal static semantics

Substitution

```
(* [subst e v x] is e{v/x}, that is,
 * [e] with [v] substituted for [x]. *)
let rec subst e v x = match e with
  | Var y -> if x=y then v else e
  | Int n -> Int n
  | Add(el,er) ->
    Add(subst el v x, subst er v x)
  | Let(y,ebind,ebody) ->
    let ebind' = subst ebind v x in
    if x=y
    then Let(y, ebind', ebody)
    else Let(y, ebind', subst ebody v x)
```

Step

```
let rec step = function
  | Int n -> failwith "Does not step"
  | Add(Int n1, Int n2) -> Int (n1 + n2)
  | Add(Int n1, e2) -> Add (Int n1, step e2)
  | Add(e1, e2) -> Add (step e1, e2)
  | Var _ -> failwith "Unbound variable"
  | Let(x, Int n, e2) -> subst e2 (Int n) x
  | Let(x, e1, e2) -> Let (x, step e1, e2)
```

FORMAL SYNTAX

Abstract syntax of expression lang.

```
e ::= x | i | e1+e2
    | let x = e1 in e2
```

`e`, **`x`**, **`i`**: *meta-variables* that stand for pieces of syntax

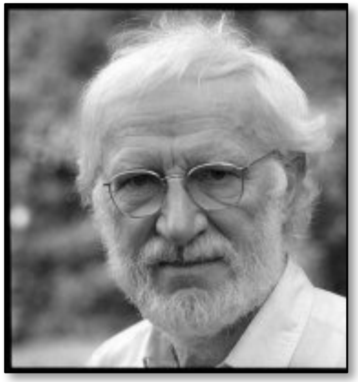
- **`e`**: expressions
- **`x`**: program variables, aka identifiers
- **`i`**: integer constants, aka literals

`::=` and | are *meta-syntax:* used to describe syntax of language

notation is called *Backus-Naur Form* (BNF) from its use by Backus and Naur in their definition of Algol-60

Backus and Naur

John Backus (1924-2007)
ACM Turing Award Winner 1977
"For profound, influential, and lasting contributions to the design of practical high-level programming systems"



Peter Naur (1928-2016)
ACM Turing Award Winner 2005
"For fundamental contributions to programming language design"

Abstract syntax of expr. lang.

```
e ::= x | i | e1+e2
    | let x = e1 in e2
```

Note resemblance of BNF to AST type:

```
type expr =
  | Var of string
  | Int of int
  | Add of expr * expr
  | Let of string * expr * expr
```

Extended with Booleans

```
e ::= x | i | b
    | e1 + e2 | e1 && e2
    | let x = e1 in e2
    | if e1 then e2 else e3

v ::= i | b
```

FORMAL DYNAMIC SEMANTICS

Dynamic semantics

Defined as a binary relation:

```
   e --> e'
```

Read as `e` takes a single step to `e'`

e.g., `(5+2)+0 --> 7+0`

Expressions continue to step until they reach a *value*

e.g., `(5+2)+0 --> 7+0 --> 7`

Values are a syntactic subset of expressions:

```
v ::= i | b
```

Dynamic semantics

Reflexive transitive closure of `-->` is written `-->*`

`e -->* e'` read as `e` multisteps to `e'`

e.g.,

```
(5+2)+0  -->*  (5+2)+0
(5+2)+0  -->*  7+0
(5+2)+0  -->*  7
```

This style of definition is called a *small-step semantics:* based on taking single small steps

Dynamic semantics of expr. lang.

`e1 + e2 --> e1' + e2`

if `e1 --> e1'`

`v1 + e2 --> v1 + e2'`

if `e2 --> e2'`

`v1 + v2 --> i`

if `i` *is the result of primitive operation* `v1+v2`

Dynamic semantics of expr. lang.

`let x = e1 in e2 --> let x = e1' in e2`

if `e1 --> e1'`

`let x = v1 in e2 --> e2{v1/x}`

recall: read **`e2{v1/x}`** as **`e2`** with **`v1`** substituted for **`x`**

(as we defined last lecture and implemented in **`subst`**)

so we call this the substitution model of evaluation

Evaluation models

- **Small-step substitution model:**
  - Substitute value for variable in body of `let` expression
  - And in body of function, since `let x = e1 in e2` behaves the same as `(fun x -> e2) e1`
  - Inefficient implementation: have to do too much substitution at run time
  - Not really what OCaml does
  - Good mental model for evaluation
- **Big-step environment model:**
  - Keep a data structure that binds variables to values
  - At the heart of what OCaml really does

Dynamic semantics of expr. lang.

```
if e1 then e2 else e3
--> if e1' then e2 else e3
```

if `e1 --> e1'`

```
if true then e2 else e3 --> e2
```

```
if false then e2 else e3 --> e3
```

Dynamic semantics of expr. lang.

Values and variables do not single step:

`v -/->` (values are already done evaluating)

`x -/->` (should have been substituted away)

But they do multistep (because they can take 0 steps):

`v -->* v`

`x -->* x`

Scaling up to OCaml

Read notes on website: full dynamic semantics for core OCaml:

```
e ::= x | e1 e2 | fun x -> e
    | i | e1 + e2
    | (e1, e2) | fst e1 | snd e2
    | Left e | Right e
    | match e with Left x -> e1 | Right y -> e2
    | let x = e1 in e2
```

Missing: other built-in types, records, lists, options, declarations, patterns in function arguments and let bindings, if expressions, recursion

FORMAL STATIC SEMANTICS

Static semantics

We can have nonsensical expressions:

```
5 + false
if 5 then true else 0
```

Need *static semantics* (type checking) to rule those out...

`if` expressions [from lec 2]

Syntax:

`if e1 then e2 else e3`

Type checking:

if **`e1`** has type **`bool`** and **`e2`** has type **`t`** and **`e3`** has type **`t`** then **`if e1 then e2 else e3`** has type **`t`**

Static semantics

Defined as a ternary relation:

`T |- e : t`

- Read as in typing context `T`, expression `e` has type `t`
- Turnstile `|-` can be read as "proves" or "shows"
- You're already used to `e : t`, because utop uses that notation
- *Typing context* is a dictionary mapping variable names to types

Static semantics

e.g.,

```
x:int |- x+2 : int
x:int,y:int |- x<y : bool
|- 5+2 : int
```

Static semantics of expr. lang.

`T |- i : int`

`T |- b : bool`

Static semantics of expr. lang.

`T |- e1 + e2 : int`
if `T |- e1 : int`
and `T |- e2 : int`

`T |- e1 && e2 : bool`
if `T |- e1 : bool`
and `T |- e2 : bool`

Static semantics of expr. lang.

```
T |- if e1 then e2 else e3 : t
   if  T |- e1 : bool
   and T |- e2 : t
   and T |- e3 : t

T |- let x:t1 = e1 in e2 : t2
   if  T |- e1 : t1
   and T, x:t1 |- e2 : t2

T, x:t |- x : t
```

To avoid need for type inference, require type annotation here

Purpose of type system

Ensure **type safety:** well-typed programs don't get *stuck:*

- haven't reached a value, and
- unable to evaluate further

Lemmas:

Progress: if `e : t`, then either `e` is a value or `e` can take a step.

Preservation: if `e : t`, and if `e` takes a step to `e'`, then `e' : t`.

Type safety = progress + preservation

Proving type safety is a fun part of CS 4110

Q: *Why bother doing proofs about programming languages? They are almost always boring if the definitions are right.*

A: *The definitions are almost always wrong.*

—Anonymous

Interpreter for expr. lang.

See **`interp3.ml`** in code for this lecture

1. Type-checks expression, then
2. Evaluates expression

Upcoming events

- [Wednesday] A3 due
- [Thursday] A4 out