Courant Differential and Integral Calculus 1959

CHAPTER I

Introduction

The differential and integral calculus is based upon two concepts of outstanding importance, apart from the concept of number, namely, the concept of *function* and the concept of *limit*. These concepts can, it is true, be recognized here and there even in the mathematics of the ancients, but it is only in modern mathematics that their essential character and significance are fully brought out. In this introductory chapter we shall attempt to explain these concepts as simply and clearly as possible.

1. The Continuum of Numbers

The question as to the real nature of numbers is one which concerns philosophers more than mathematicians, and philosophers have been much occupied with it. But mathematics must be carefully kept free from conflicting philosophical opinions; preliminary study of the essential nature of the concept of number from the point of view of the theory of knowledge is fortunately not required by the student of mathematics. We shall therefore take the numbers, and in the first place the natural numbers 1, 2, 3, . . ., as given, and we shall likewise take as given the rules* by which we calculate with these numbers; and we shall only briefly recall the way in which the concept of the positive integers (the natural numbers) has had to be extended.

* These rules are as follows: $(a + b) + c = a + (b + c)$. That is, if to the sum of two numbers a and b we add a third number c, we obtain the same result as when we add to a the sum of b and c. (This is called the associative law of addition.) Secondly, $a + b = b + a$ (the commutative law of addition). Thirdly, $(ab)c = a(bc)$ (the associative law of multiplication). Fourthly, $ab = ba$ (the commutative law of multiplication). Fifthly, $a(b + c) = ab + ac$ (the distributive law of multiplication).

1. The System of Rational Numbers and the Need for its Extension.

In the domain of the natural numbers the fundamental operations of addition and multiplication can always be performed without restriction; that is, the sum and the product of two natural numbers are themselves always natural numbers. But the inverses of these operations, subtraction and division, cannot invariably be performed within the domain of natural numbers; and because of this mathematicians were long ago obliged to invent the number 0, the negative integers, and positive and negative fractions. The totality of all these numbers is usually called the class of *rational numbers*, since they are all obtained from unity by using the " rational operations of calculation ", addition, multiplication, subtraction and division.

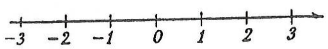

Fig. 1.—The number axis

Numbers are usually represented graphically by means of the points of a straight line, the " number axis ", by taking an arbitrary point of the line as the origin or zero point and another point of the line as the point 1; the distance between these two arbitrary points (the length of the *unit interval*) then serves as a scale by which we can assign a point on the line to every rational number, positive or negative. It is customary to mark off the positive numbers to the right and the negative numbers to the left of the origin (cf. fig. 1). If, as usual, we define the absolute value (also called the numerical value or modulus) $|a|$ of a number a to be a itself when* $a \geqq 0$, and to be $-a$ when $a < 0$, then $|a|$ simply denotes the distance of the corresponding point on the number axis from the origin.

The geometrical representation of the rational numbers by points on the number axis suggests an important property which is usually stated as follows: *the set of rational numbers is everywhere dense.* This means that in every interval of the number axis, no matter how small, there are always rational numbers; geometrically, in the segment of the number axis between any two rational points, no matter how close together, there are points corresponding to rational numbers. This density of the rational

* By the sign $\geqq$ we mean that *either* the sign $>$ *or* the sign $=$ shall hold. A corresponding statement holds for the signs $\pm$ and $\mp$ which will be used later.

numbers at once becomes clear if we start from the fact that the numbers $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots$ become steadily smaller and approach nearer and nearer to zero as n increases. If we now divide the number axis into equal parts of length $1/2^n$, beginning at the origin, the end-points $\frac{1}{2^n}, \frac{2}{2^n}, \frac{3}{2^n}, \ldots$ of these intervals represent rational numbers of the form $m/2^n$; here we still have the number n at our disposal. If now we are given a fixed interval of the number axis, no matter how small, we need only choose n so large that $1/2^n$ is less than the length of the interval; the intervals of the above subdivision are then small enough for us to be sure that at least one of the points of subdivision $m/2^n$ lies in the interval.

Yet in spite of this property of density the rational numbers are not sufficient to represent *every* point on the number axis. Even the Greek mathematicians recognized that when a given line segment of unit length is chosen there are intervals whose lengths cannot be represented by rational numbers; these are the so-called segments incommensurable with the unit. Thus, for example, the hypotenuse of a right-angled isosceles triangle with sides of unit length is not commensurable with the unit of length. For, by the theorem of Pythagoras, the square of this length l must be equal to 2. Therefore, if l were a rational number and consequently equal to p/q, where p and q are integers different from 0, we should have $p^2 = 2q^2$. We can assume that p and q have no common factors, for such common factors could be cancelled out to begin with. Since, according to the above equation, p^2 is an even number, p itself must be even, say $p = 2p'$. Substituting this expression for p gives us $4p'^2 = 2q^2$, or $q^2 = 2p'^2$; consequently q^2 is even, and so q is also even. Hence p and q both have the factor 2. But this contradicts our hypothesis that p and q have no common factor. Thus the assumption that the hypotenuse can be represented by a fraction p/q leads to contradiction and is therefore false.

The above reasoning, which is a characteristic example of an " indirect proof ", shows that the symbol $\sqrt{2}$ cannot correspond to any rational number. Thus we see that if we insist that after choice of a unit interval every point of the number axis shall have a number corresponding to it, we are forced to extend

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the domain of rational numbers by the introduction of new "irrational" numbers. This system of rational and irrational numbers, such that each point on the axis corresponds to just one number and each number corresponds to just one point on the axis, is called the system of *real numbers*.*

2. Real Numbers and Infinite Decimals.

Our requirement that to each point of the axis there shall correspond one real number states nothing *a priori* about the possibility of calculating with these real numbers in the same way as with rational numbers. We establish our right to do this by showing that our requirement is equivalent to the following fact: the totality of all real numbers is represented by the totality of all finite and infinite decimals.

We first recall the fact, familiar from elementary mathematics, that every rational number can be represented by a terminating or by a recurring decimal; and conversely, that every such decimal represents a rational number. We shall now show that to *every* point of the number axis we can assign a uniquely determined decimal (usually infinite), so that we can represent the irrational points or irrational numbers by infinite decimals. (In accordance with the above remark the irrational numbers must be represented by infinite non-recurring decimals, for example, 0·101101110 . . .).

Suppose that the points which correspond to the integers are marked on the number axis. By means of these points the axis is subdivided into intervals or segments of length 1. In what follows, we shall say that a point of the line belongs to an interval if it is an interior point or an end-point of the interval. Now let P be an arbitrary point of the number axis. Then the point belongs to one, or if it is a point of division to two, of the above intervals. If we agree that in the second case the right-hand one of the two intervals meeting at P is to be chosen, we have in all cases an interval with end-points g and $g + 1$ to which P belongs, where g is an integer. This interval we subdivide into ten equal sub-intervals by means of the points corresponding to the numbers $g + \frac{1}{10}, g + \frac{2}{10}, \ldots, g + \frac{9}{10}$, and

* Thus named to distinguish it from the system of complex numbers, obtained by yet another extension.

we number these sub-intervals 0, 1, . . . , 9 in the natural order from left to right. The sub-interval with the number a then has the end-points $g + \frac{a}{10}$ and $g + \frac{a}{10} + \frac{1}{10}$. The point P must be contained in one of these sub-intervals. (If P is one of the new points of division it belongs to two consecutive intervals; as before, we choose the one on the right.) Suppose that the interval thus determined is associated with the number a_1. The end-points of this interval then correspond to the numbers $g + \frac{a_1}{10}$ and $g + \frac{a_1}{10} + \frac{1}{10}$. This sub-interval we again divide into ten equal parts and determine that one to which P belongs; as before, if P belongs to two sub-intervals we choose the one on the right. We thus obtain an interval with the end-points $g + \frac{a_1}{10} + \frac{a_2}{10^2}$ and $g + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{1}{10^2}$, where a_2 is one of the digits 0, 1, . . . , 9. This sub-interval we again subdivide, and continue to repeat the process. After n steps we arrive at a sub-interval containing P, having length $\frac{1}{10^n}$ and with end-points corresponding to the numbers

$$g + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} \quad \text{and} \quad g + \frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n} + \frac{1}{10^n}.$$

Here each a is one of the numbers 0, 1, . . . , 9. But

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \ldots + \frac{a_n}{10^n}$$

is simply the decimal fraction $0{\cdot}a_1a_2 \ldots a_n$. The end-points of the interval, therefore, may also be written in the form

$$g + 0{\cdot}a_1a_2 \ldots a_n \quad \text{and} \quad g + 0{\cdot}a_1a_2 \ldots a_n + \frac{1}{10^n}.$$

If we consider the above process repeated indefinitely, we obtain an *infinite decimal* $0{\cdot}a_1a_2 \ldots$, which has the following meaning. If we break off this decimal at any place, say the n-th, the point P will lie in the interval of length $\frac{1}{10^n}$ whose end-points (approximating points) are

$$g + 0{\cdot}a_1a_2 \ldots a_n \quad \text{and} \quad g + 0{\cdot}a_1a_2 \ldots a_n + \frac{1}{10^n}.$$

In particular, the point corresponding to the rational number $g + 0{\cdot}a_1a_2 \ldots a_n$ will lie arbitrarily near to the point P if only n is large enough; for this reason the points $g + 0{\cdot}a_1a_2 \ldots a_n$ are called approximating points. *We say that the infinite decimal* $g + 0{\cdot}a_1a_2 \ldots$ *is the real number corresponding to the point* P.

Here we would emphasize the fundamental *assumption* that we can calculate in the usual way with the real numbers, and hence with the decimals. It is possible to prove this using only the properties of the integers as a starting-point. But this is no light task; and rather than allow it to bar our progress at this early stage, we regard the fact that the ordinary rules of calculation apply to the real numbers as an axiom, on which we shall base the whole differential and integral calculus.

We here insert a remark concerning the possibility, in certain cases, of choosing the interval in *two* ways in the above scheme of expansion. From our construction it follows that the points of division arising in our repeated process of subdivision, and such points only, can be represented by finite decimals $g + 0{\cdot}a_1a_2 \ldots a_n$. Let us suppose that such a point P first appears as a point of division at the n-th stage of the subdivision. Then according to the above process we have chosen at the n-th stage the interval to the right of P. In the following stages we must choose a sub-interval of this interval. But such an interval must have P as its left end-point. Therefore in all further stages of the subdivision we must choose the first sub-interval, which has the number 0. Thus the infinite decimal corresponding to P is $g + 0{\cdot}a_1a_2 \ldots a_n000 \ldots$. If, on the other hand, we had at the n-th stage chosen the left-hand interval containing P, then in all later stages of subdivision we should have had to choose the sub-interval farthest to the right, which has P as its right end-point. Such a sub-interval has the number 9. Thus for P we should have obtained a decimal expansion in which all the digits from the $(n + 1)$-th onward are nines. The double possibility of choice in our construction therefore corresponds to the fact that for example the number $\frac{1}{4}$ has the two decimal expansions $0{\cdot}25000 \ldots$ and $0{\cdot}24999 \ldots$.

3. Expression of Numbers in Scales other than that of 10.

In our representation of the real numbers we made the number 10 play a special part, for each interval was subdivided into ten equal parts. The only reason for this is the widespread use of the decimal system. We could just as well have taken p equal sub-intervals, where p is an arbitrary integer greater than 1. We should then have obtained an expression of the form $g + \frac{b_1}{p} + \frac{b_2}{p^2} + \ldots$, where each b is one of the numbers

$0, 1, \ldots, p - 1$. Here again we find that the rational numbers, and only the rational numbers, have recurring or terminating expansions of this kind. For theoretical purposes it is often convenient to choose $p = 2$. We then obtain the *binary expansion* of the real numbers,

$$g + \frac{b_1}{2} + \frac{b_2}{2^2} + \ldots,$$

where each b is either * 0 or 1.

For numerical calculations it is customary to express the whole number g, which for simplicity we here take to be positive, in the decimal system, that is, in the form

$$a_m 10^m + a_{m-1}10^{m-1} + \ldots + a_1 10 + a_0,$$

where each a_ν is one of the digits $0, 1, \ldots, 9$. Then for $g + 0{\cdot}a_1a_2 \ldots$ we write simply

$$a_m a_{m-1} \ldots a_1 a_0 \cdot a_1 a_2 \ldots$$

Similarly, the positive whole number g can be written in one and only one way in the form

$$\beta_k p^k + \beta_{k-1}p^{k-1} + \ldots + \beta_1 p + \beta_0,$$

where each of the numbers β_ν is one of the numbers $0, 1, \ldots, p - 1$. This, with our previous expression, gives the following result: every positive real number can be represented in the form

$$\beta_k p^k + \beta_{k-1}p^{k-1} + \ldots + \beta_1 p + \beta_0 + \frac{b_1}{p} + \frac{b_2}{p^2} + \ldots,$$

where β_ν and b_ν are whole numbers between 0 and $p - 1$. Thus, for example, the binary expansion of the fraction $\frac{21}{4}$ is

$$\frac{21}{4} = 1 \times 2^2 + 0 \times 2 + 1 + \frac{0}{2} + \frac{1}{2^2}.$$

* Even for numerical calculations the decimal system is not the best. The sexagesimal system ($p = 60$), with which the Babylonians calculated, has the advantage that a comparatively large proportion of the rational numbers whose decimal expansions do not terminate possess terminating sexagesimal expansions.