Behavioral Equivalence

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Today’s music: *Soul Bossa Nova* by Quincy Jones
Review

Previously in 3110:
• Functional programming
• Modular programming & software engineering
• Interpreters

Today:
• Reasoning about correctness of programs
Building Reliable Software

• Suppose you work at (or run) a software company.

• Suppose you’ve sunk 30+ person-years into developing the “next big thing”:
  – Boeing Dreamliner2 flight controller
  – Autonomous vehicle control software for Nissan
  – Gene therapy DNA tailoring algorithms
  – Super-efficient green-energy power grid controller

• How do you avoid disasters?
  – Turns out software endangers lives
  – Turns out to be impossible to build software
Approaches to Reliability

• Social
  – Code reviews
  – Extreme/Pair programming

• Methodological
  – Design patterns
  – Test-driven development
  – Version control
  – Bug tracking

• Technological
  – Static analysis ("lint" tools, FindBugs, …)
  – Fuzzers

• Mathematical
  – Sound type systems
  – “Formal” verification

Less formal: Techniques may miss problems in programs

All of these methods should be used!

Even the most formal can still have holes:
• did you prove the right thing?
• do your assumptions match reality?

More formal: eliminate with certainty as many problems as possible.
Testing vs. Verification

Testing:
• Cost effective
• Guarantee that program is correct on tested inputs and in tested environments

Verification:
• Expensive
• Guarantee that program is correct on all inputs and in all environments
Edsger W. Dijkstra

Turing Award Winner (1972)

*For eloquent insistence and practical demonstration that programs should be composed correctly, not just debugged into correctness*

"Program testing can at best show the presence of errors but never their absence."

(1930-2002)
Verification

• In the 1970s, scaled to about tens of LOC
• Now, research projects scale to real software:
  – **CompCert**: verified C compiler
  – **seL4**: verified microkernel OS
  – **Ynot**: verified DBMS, web services
• In another 40 years?
Our trajectory

- Proofs about functions
- Proofs about variants
- Proofs about modules

- We’re not trying to get all the way to fully machine-checked correctness proofs of large programs
- Rather:
  - help you understand what it means to be correct
  - help you organize your thoughts about correctness of code you write

- Important caveat: no side-effects! Specifically, no mutability or I/O.
Example

```ocaml
let rec even n =
  match n with
  | 0 -> true
  | 1 -> false
  | n -> even (n-2)
```

**Theorem.** For all natural numbers $n$, it holds that $even \ (2* n)$ is true.
Example

(* precondition: \( n \geq 0 \) *)
(* postcondition: \( (\text{fact } n) = n! \) *)

\[
\text{let rec fact } n = \\
\text{ if } n=0 \text{ then } 1 \\
\text{ else } n \times \text{fact } (n-1)
\]

Theorem. \text{fact} is correct—it satisfies its specification.
Example

```plaintext
let rec length = function
  | []  -> 0
  | _::xs -> 1 + length xs

let rec append xs1 xs2 = match xs1 with
  | []  -> xs2
  | h::t -> h :: append t xs2
```

**Theorem.** For all lists `xs` and `ys`, it holds that `length (append xs ys) is length xs + length ys`.
EQUIVALENCE OF EXPRESSIONS
Behavioral equivalence

• Behavioral equivalence: two expressions behave the same
  – always evaluate to same value?
Question

Which of these expressions is behaviorally equivalent to 42?

A. if b then 42 else 42
   (for an arbitrary Boolean expression b)
B. let _ = f x in 42
   (for an arbitrary function f and argument x)
C. List.hd [42]
D. All of the above
E. None of the above
Question

Which of these expressions is behaviorally equivalent to 42?

A. `if b then 42 else 42`
   (for an arbitrary Boolean expression b)
B. `let _ = f x in 42`
   (for an arbitrary function f and argument x)
C. `List.hd [42]`
D. All of the above
E. None of the above
Behavioral equivalence

• **Behavioral equivalence:** two expressions behave the same
  – always evaluate to same value
  – (or always raise the same exception)
  – (or always diverge: don't terminate)

• Write as $e_1 \sim e_2$
  – I would much prefer $e_1 \equiv e_2$, but that symbol isn't available in plain text
Behavioral equivalence

Fundamental axioms about when expressions are behaviorally equivalent:

• **eval**: if $e_1 \rightarrow^* e_2$ then $e_1 \sim e_2$

• **alpha**: if $e_1$ differs from $e_2$ only by consistent renaming of variables then $e_1 \sim e_2$

• **sugar**: if $e_1$ is syntactic sugar for $e_2$ then $e_1 \sim e_2$
Behavioral equivalence

Facts (theorems) about behavioral equivalence:

- **reflexive**: $e \sim e$
- **symmetric**: if $e_1 \sim e_2$ then $e_2 \sim e_1$
- **transitive**: if $e_1 \sim e_2$ and $e_2 \sim e_3$ then $e_1 \sim e_3$

...that is, $\sim$ is an *equivalence relation*
Easy example with ~

\texttt{let} easy x y z = x \ast (y + z)

Theorem: easy 1 20 30 \sim 50

Proof:

\hspace{1em} easy 1 20 30
\sim 50 \hspace{1em} (by eval)

QED
Another easy example

\textbf{let} easy x y z = x * (y + z)

Theorem:
for all ints n and m, easy 1 n m \sim n + m

Proof:
  easy 1 n m
\sim n + m   \text{(by eval)}
QED

Not so!
Evaluation with unknown values

• That proof wasn't valid according to the small-step semantics:
  – `easy 1 n m ->`
  – because `n` and `m` aren't strictly speaking values
  – they might as well be, though...

• *Symbolic values:* they stand for a value
  – Think of them as "mathematical variables" as opposed to "program variables"
  – They are values; we just don't know what they are
  – We'll allow the semantics to consider them as values

• So we can allow evaluation to continue:
  – `easy 1 n m -> x*(y+z){1/x}{n/y}{m/z} -> 1*(n+m) ->`
  – because `n+m` isn't strictly speaking a value
  – it might as well be, though; guaranteed to produce a value at runtime...
Valuable expressions

• *Valuable*: guaranteed to produce a value
  – No exceptions
  – Always terminates

• If an expression is valuable, then *we may use it as though it were already a value* in the semantics

• So we can allow evaluation to continue:
  
  easy 1 n m
  -> x*(y+z){1/x}{n/y}{m/z}
  -> 1*(n+m)
  -> n+m
Valuable expressions

Definition of *valuable*:

- a (symbolic) value is valuable
- a variable is valuable: at run-time, will be replaced by a value
- a constant is valuable
- any pair, record, or variant built out of valuable expressions is valuable
- an `if` expression is valuable if all its subexpressions are valuable
- a `match` expression is valuable if its subexpressions are valuable and it is exhaustive: non-exhaustive could raise exception at run time
- a function application is valuable if the argument is valuable and the function is *total*: guaranteed to terminate with a value
  - `+` is total
  - `/` is *partial*, as is `List.hd`, so their application is not necessarily valuable
- a `raise` expression is never valuable
- a `try` expression is valuable if its handler expressions are valuable, its main expression terminates, and all exceptions that could be raised by the main expression have a handler
Why we need totality

\begin{verbatim}
let rec forever x = forever ()
let one x = 1
\end{verbatim}

If we didn't require functions to be total, we would conclude

\[
\text{one (forever ()) \to 1_{\text{forever()}/x} = 1}
\]

hence

\[
\text{one (forever ()) \sim 1}
\]

which violates the definition of behavioral equivalence
Why we need totality (again)

\[
\text{let one x = 1}
\]

If we didn't require functions to be total, we would conclude

\[
\text{one (List.hd [])} \\
\rightarrow 1\{\text{List.hd []}/x\} = 1
\]

hence

\[
\text{one (List.hd [])} \sim 1
\]

which violates the definition of behavioral equivalence
**Using valuable expressions**

```plaintext
let easy x y z = x * (y + z)
```

**Theorem:** for all ints a, b, and c,
```plaintext
easy a b c ~ easy a c b
```

**Proof:**
```plaintext
easy a b c
~ a * (b + c)   (by eval)
~ a * (c + b)   (???)
~ easy a c b   (by eval, symm.)
QED
```
"By math"

Assume basic algebraic properties of the OCaml built-in operators:

- \((r+s)+t \sim r + (s + t)\)
- \(r+s \sim s+r\)
- \(r+0 \sim 0+r \sim r\)
- \(r + (-r) \sim 0\)
- \(r*s \sim s*r\)
- \((r*s)*t \sim r*(s*t)\)
- \(r*0 \sim 0*r \sim 0\)
- \(r*1 \sim 1*r \sim r\)
- \(r*(s+t) \sim (r*s)+(r*t)\)
- \((r+s)*t \sim (r*t)+(s*t)\)
- etc.

where \(r, s, t\) must be valuable
"By math"

Allow use of other mathematical operators that aren't built-in to OCaml:
• Integer exponentiation
• Factorial
• etc.

All arguments must be valuable

e.g.
(k+1)! \sim (k+1)*(k!) \ (by\ math,\ if\ k\geq0)
Using valuable expressions

```plaintext
let easy x y z = x * (y + z)
```

Theorem: for all ints a, b, and c, 
`easy a b c ~ easy a c b`

Proof:
- `easy a b c`
- `~ a * (b + c)` (by eval)
- `~ a * (c + b)` (by math)
- `~ easy a c b` (by eval, symm.)
QED
Doubles are even

(* requires:  n >= 0 *)

let rec even n =
  match n with
  | 0 -> true
  | 1 -> false
  | n -> even (n-2)

Theorem:
for all natural numbers n,
even (2*n) ~ true.

Naturals: integers >= 0
We ignore the limits of machine arithmetic here.
Doubles are even

Theorem:
for all natural numbers n, even \((2\times n) \sim \) true.

Proof: by induction. QED

A PL theorist's favorite proof. :)
Review: Induction on natural numbers

Theorem: 
for all natural numbers \( n \), \( P(n) \).

Proof: by induction on \( n \)

Case: \( n \) is 0 
Show: \( P(0) \)

Case: \( n \) is \( k+1 \) 
IH: \( P(k) \)
Show: \( P(k+1) \)

QED
Induction principle

for all properties \( P \) of natural numbers,
if \( P \ 0 \)
and (for all \( n \), \( P\ n \) implies \( P\ (n+1) \))
then (for all \( n \), \( P\ n \))
Doubles are even

Theorem:
for all natural numbers \( n \), even \((2*n)\) ~ true.

Proof: by induction on \( n \)

Case: \( n \) is 0
Show: even \((2*0)\) ~ true

\[
\text{even } (2*0) \\
\sim \text{true} \quad \text{(eval)}
\]
Doubles are even

Theorem:
for all natural numbers \( n \), even \( (2*n) \) ~ true.

Proof: by induction on \( n \)

Case: \( n \) is \( k+1 \), where \( k \geq 0 \)
IH: even \( (2*k) \) ~ true
Show: even \( (2*(k+1)) \) ~ true

\[
\begin{align*}
\text{let rec } \ & \text{even } n = \\
& \text{match } n \text{ with} \\
& 0 \rightarrow \text{true} \\
& 1 \rightarrow \text{false} \\
& n \rightarrow \text{even } (n-2)
\end{align*}
\]
Question

What would justify this proof step?

\[ \text{even } (2 \cdot (k+1)) \sim \text{even } (2 \cdot k + 2) \]

A. math
B. eval
C. transitivity
D. All the above together
E. None of the above
Question

What would justify this proof step?

\[ \text{even } (2 \times (k+1)) \sim \text{even } (2 \times k+2) \]

A. math
B. eval
C. transitivity
D. All the above together
E. None of the above
**Congruence**

A deep fact about behavioral equivalence:

**congruence:**

\[
\text{if } e_1 \sim e_2 \text{ then } e\{e_1/x\} \sim e\{e_2/x\}
\]

aka *substitution of equals for equals* and *Leibniz equality*

Congruence is hugely important: enables local reasoning
- replace small part of large program with an equivalent small part
- conclude equivalence of large programs without having to do large proof!
Doubles are even

Theorem:
for all natural numbers \( n \), even \( 2n \) ~ true.

Proof: by induction on \( n \)

Case: \( n = k+1 \), where \( k \geq 0 \)

IH: even \( 2k \) ~ true

Show: even \( 2(k+1) \) ~ true

\[
\begin{align*}
\text{even } (2(k+1)) \\
\sim \text{even } (2k+2) & \quad \text{(math, congr.)} \\
\sim \text{even } (2k+2-2) & \quad \text{(eval, } k \geq 0) \\
\sim \text{even } (2k) & \quad \text{(math, congr.)} \\
\sim \text{true} & \quad \text{(IH)}
\end{align*}
\]

QED
Upcoming events

• [Wed] A4 due
• [following Wed] MS1 due

This is well behaved.

THIS IS 3110
Acknowledgements

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Academic genealogy: Constable -> Harper -> Morrisett -> Walker (-> means advised)