CS 3110

Efficiency

Prof. Clarkson
Fall 2016

Today's music: Opening theme from *The Big O*
(THE ビッグオ)
by Toshihiko Sahashi

Review

Previously in 3110:

- Functional programming
- Modular programming and software engineering
- Interpreters

Today:

- Interlude on efficiency of programs

Question

Which of the following would you prefer?

A. O(n^2)
B. O($log(n)$)
C. O(n)
D. They're all good
E. I thought this was 3110, not Algo

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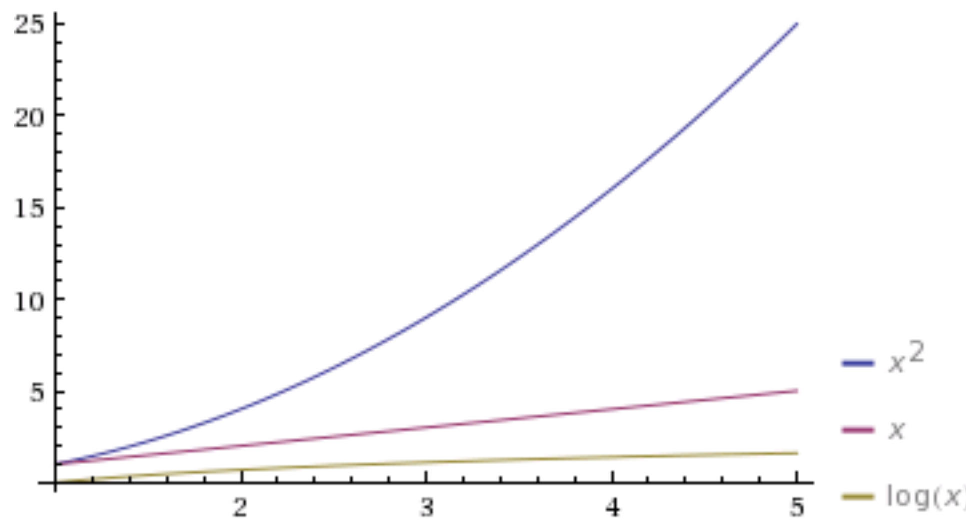

What is "efficiency"?

Attempt #1: An algorithm is efficient if, when implemented, it runs in a small amount of time on particular input instances

...problems with that?

What is "efficiency"?

Attempt #1: An algorithm is efficient if, when implemented, it runs in a small amount of time on particular input instances

Incomplete list of problems:

- Inefficient algorithms can run quickly on small test cases
- Fast processors and optimizing compilers can make inefficient algorithms run quickly
- Efficient algorithms can run slowly when coded sloppily
- Some input instances are harder than others
- Efficiency on small inputs doesn't imply efficiency on large inputs
- Some clients can afford to be more patient than others; quick for me might be slow for you

Lessons learned from attempt #1

Lesson 1: Time as measured by a clock is not the right metric

- Want a metric that is reasonably independent of hardware, compiler, other software running, etc.
- **Idea:** number of steps taken (say, by small-step semantics) during evaluation of program
  - steps are independent of implementation details
  - but: each step might really take a different amount of time?
    - creating a closure, looking up a variable, computing an addition
  - in practice, the difference isn't really big enough to matter

Lessons learned from attempt #1

Lesson 2: Running time on particular input instances is not the right metric

- Want a metric that can predict running time on **any** input instance
- **Idea:** size of the input instance
  - make metric be a function of input size
  - (combined with lesson 1) specifically, the maximum number of steps for an input of that size
  - But: particular inputs of the same size might really take a different amount of time?
    - multiplying arbitrary matrices vs. multiplying by all zeros
  - in practice, size matters more

Lessons learned from attempt #1

Lesson 3: "Small" is too relative

- Want a metric that is reasonably objective; independent of subjective notions of what is fast
- **Okay idea:** beats brute-force search
  - *brute force*: enumerate all the answers one by one, check and see whether the answer is right
    - the simple, dumb solution to nearly any algorithmic problem
    - related idea: guess an answer, check whether correct
      e.g., bogosort
  - but *by how much* is enough to beat brute-force search?

Lessons learned from attempt #1

Lesson 3: "Small" is too relative

- **Better idea:** polynomial time
  - (combined with ideas from previous two lessons) can express maximum number of steps as a polynomial function of the size N of input, e.g.,
    - $aN^2 + bN + c$
  - But: some polynomials might be too big to be quick? e.g. N^{100}
  - But: some non-polynomials might be quick enough? e.g. $N^{1+.02(\log N)}$
  - in practice, polynomial time really does work

What is "efficiency"?

Attempt #2: An algorithm is efficient if its maximum number of steps of execution is polynomial in the size of its input.

let's give that a try...

Analysis of running time

INSERTION-SORT(A)	*cost*	*times*
for j = 2 **to** **A.length**	c_1	n
key = *A*[*j*]	c_2	$n - 1$
// Insert *A*[*j*] into the sorted sequence *A*[1 .. *j* - 1]	0	$n - 1$
i = *j* - 1	c_4	$n - 1$
while i > 0 and *A*[*i*] < *key*	c_5	$\sum_{j=2}^{n} t_j$
A[*i* + 1] = *A*[*i*]	c_6	$\sum_{j=2}^{n} (t_j - 1)$
i = *i* - 1	c_7	$\sum_{j=2}^{n} (t_j - 1)$
A[*i* + 1] = *key*	c_8	$n - 1$

[Cormen et al. *Introduction to Algorithms*, 3rd ed, 2009]

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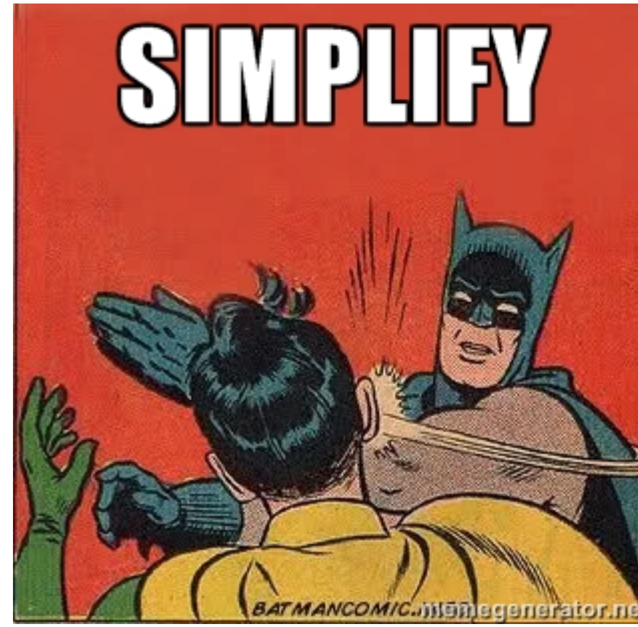

The running time of the algorithm is the sum of running times for each statement executed; a statement that takes c_i steps to execute and executes n times will contribute $c_i n$ to the total running time.[6] To compute $T(n)$, the running time of INSERTION-SORT on an input of n values, we sum the products of the *cost* and *times* columns, obtaining

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) .$$

[Cormen et al. *Introduction to Algorithms*, 3rd ed, 2009]

Precision of running time

- Precise bounds are exhausting to find
- Precise bounds are to some extent meaningless
  - Are those constants c1..c8 really useful?
  - If it takes 25 steps in high level language, but compiled down to assembly would take 10x more steps, is the precision useful?
  - Caveat: if you're building code that flies an airplane or controls a nuclear reactor, you do care about precise, real-time guarantees

Some simplified running times

max # steps as function of N

size of input

	N	N^2	N^3	2^N
N=10	< 1 sec	< 1 sec	< 1 sec	< 1 sec
N=100	< 1 sec	< 1 sec	1 sec	10^{17} years
N=1,000	< 1 sec	1 sec	18 min	very long
N=10,000	< 1 sec	2 min	12 days	very long
N=100,000	< 1 sec	3 hours	32 years	very long
N=1,000,000	1 sec	12 days	10^4 years	very long

assuming 1 microsecond/step

very long = more years than the estimated number of atoms in universe

Simplifying running times

- Rather than $1.62N^2 + 3.5N + 8$ steps, we would rather say that running time "grows like N^2"
  - identify broad classes of algorithm with similar performance
- Ignore the *low-order terms*
  - e.g., ignore $3.5N+8$
  - Why? For big N, N^2 is much, much bigger than N
- Ignore the *constant factor* of high-order term
  - e.g., ignore 1.62
  - Why? For classifying algorithms, constants aren't meaningful
    - Code run on my machine might be a constant factor faster or slower than on your machine, but that's not a property of the algorithm
  - Caveat: Performance tuning real-world code actually can be about getting the constants to be small!
- Abstraction to an imprecise quantity

Imprecise abstractions

- OCaml's **`int`** type is an abstraction of a subset of Z
  - don't know which int when reasoning about the type of an expression
- ±1 is an abstraction of {1,-1}
  - don't know which when manipulating it in a formula
- Here's a new one: Big Ell
  - L(n) represents a natural number whose value is less than or equal to n
  - precisely, $L(n) = \{m \mid 0 \leq m \leq n\}$
  - e.g., $L(5) = \{0, 1, 2, 3, 4, 5\}$

Manipulating Big Ell

- What is 1 + L(5)?
- Trick question!
  - Replace L(5) with set: 1 + {0..5}
  - But + is defined on ints, not sets of ints
- We could distribute the + over the set: {1+0, ..., 1+5} = {1..6}
  - That is, a set of values, one for each possible instantiation of L(5)
- Note that {1..6} ⊆ {0..6} = L(6)
- So we could say that 1 + L(5) ⊆ L(6)

Question

What is L(2) + L(3)?

Hint: set of values, one for each possible instantiation of L(2) and of L(3)

A. $L(2) + L(3) \subseteq L(2)$

B. $L(2) + L(3) \subseteq L(3)$

C. $L(2) + L(3) \subseteq L(4)$

D. $L(2) + L(3) \subseteq L(5)$

E. $L(2) + L(3) \subseteq L(6)$

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What is L(2) * L(3)?

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D. L(2) * L(3) ⊆ L(5)
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A little trickier...

What is $2^{L(3)}$?

- $L(3) = \{0..3\}$
- So $2^{L(3)}$ could be any of $\{2^0, \ldots, 2^3\} = \{1, 2, 4, 8\}$
- And $\{1,2,4,8\} \subseteq L(8) = L(2^3)$
- Therefore $2^{L(3)} \subseteq L(2^3)$

...we can use this idea of Big Ell to invent an imprecise abstraction for running times

Big Oh, version 1

- **Recall:** we're interested in running time as a function of input size
- **Recall:** L(*n*) represents any natural number that is less than or equal to a natural number *n*
- "New" imprecise abstraction: Big Oh
  - **Intuition:** O(*g*) represents any function that is less than or equal to function *g*, for every input *n*
  - Big Oh is a higher-order version of Big Ell: generalize from naturals to functions on naturals
- Why the naturals? We're assuming function inputs and outputs are non-negative:
  - These are functions on input size and running time
  - Those won't be negative

Big Oh, version 1

Definition: $O(g) = \{ f \mid \forall n \,.\, f(n) \le g(n) \}$

e.g.

- $O(fun\ n \rightarrow 2n) = \{f \mid \forall n \,.\, f(n) \le 2n\}$
- $(fun\ n \rightarrow n) \in O(fun\ n \rightarrow 2n)$

Note: these are mathematical functions written in OCaml notation, not OCaml functions

Big Oh, version 2

Recall: we want to ignore constant factors

$(fun\ n \rightarrow n)$, $(fun\ n \rightarrow 2n)$, $(fun\ n \rightarrow 3n)$

...all should be in O$(fun\ n \rightarrow n)$

Revised intuition: O(g) represents any function that is less than or equal to function g times some positive constant c, for every input n

Big Oh, version 2

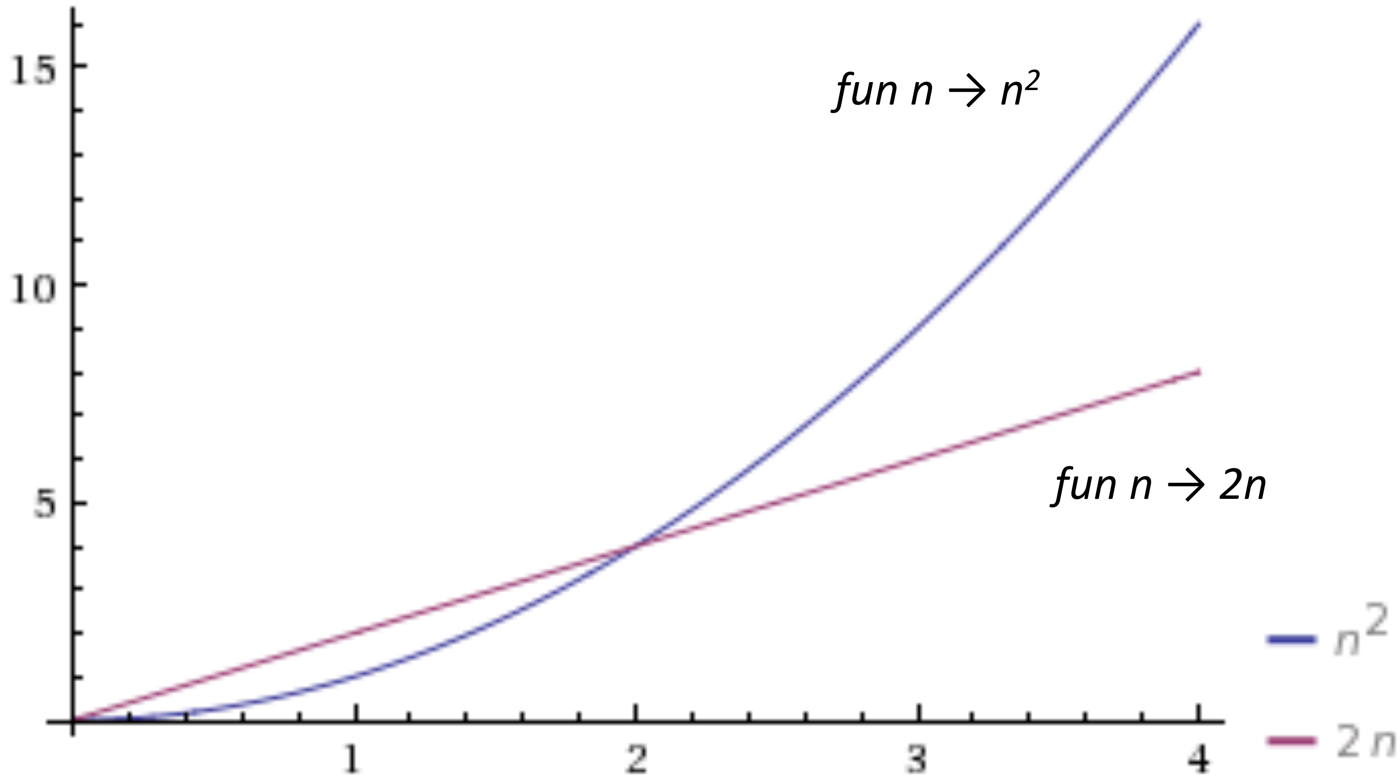
Definition: $O(g) = \{ f \mid \exists c>0 \,.\, \forall n \,.\, f(n) \leq c\, g(n) \}$

e.g.

- $O(fun\ n \rightarrow n^3) = \{ f \mid \exists c>0\ \forall n \,.\, f(n) \leq cn^3 \}$
- $(fun\ n \rightarrow 3n^3) \in O(fun\ n \rightarrow n^3)$
  because $3n^3 \leq cn^3$, where $c = 3$ (or $c=4$, ...)

Big Oh, version 3

Recall: we care about what happens at scale



could just build a lookup table for inputs in the range 0..2

Big Oh, version 3

Recall: we care about what happens at scale

Revised intuition: O(*g*) represents any function that is less than or equal to function *g* times some positive constant *c*, for every input *n* greater than or equal to some positive constant n_0

Big Oh, version 3

Definition:

$O(g) = \{ f \mid \exists\, c>0, n_0>0 \,.\, \forall\, n \geq n_0 \,.\, f(n) \leq c\, g(n) \}$

this is the important, final definition you should know!

e.g.:

- $O(fun\ n \rightarrow n^2) = \{ f \mid \exists\, c>0, n_0>0 \,.\, \forall\, n \geq n0 \,.\, f(n) \leq cn^2 \}$
- $(fun\ n \rightarrow 2n) \in O(fun\ n \rightarrow n^2)$
  because $2n \leq cn^2$, where $c = 2$, for all $n \geq 1$

Big Oh Notation: Warning 1

Instead of

$O(g) = \{f \mid ...$

most authors write

$O(g(n)) = \{f(n) \mid ...$

- They don't really mean *g* applied to *n*; they mean a function *g* parameterized on input *n* but not yet applied
- Maybe they never studied functional programming ☺

Big Oh Notation: Warning 2

Instead of

$(fun\ n \rightarrow 2n) \in O(fun\ n \rightarrow n^2)$

all authors write

$2n = O(n^2)$

- Your instructor has always found this abusage distressing...
- Yet henceforth he will conform to the convention ☺
- The standard defense is that = should be read here as "is" not as "equals"
- Be careful: one-directional "equality"!

A Theory of Big Oh

- reflexivity: $f = O(f)$
- *(no symmetry condition for Big Oh)*
- transitivity: if $f = O(g)$ and $g = O(h)$ then $f = O(h)$
- $c\, O(f) = O(f)$
- $O(c\, f) = O(f)$
- $O(f)\, O(g) = O(f\, g)$
  where $f\, g$ means $(\mathit{fun}\ n \rightarrow f(n)\ g(n))$

Useful to know these equalities so that you don't have to keep re-deriving them from first principles

What is "efficiency"?

Final attempt: An algorithm is efficient if its worst-case running time on input size N is $O(N^d)$ for some constant d.

Running times of some algorithms

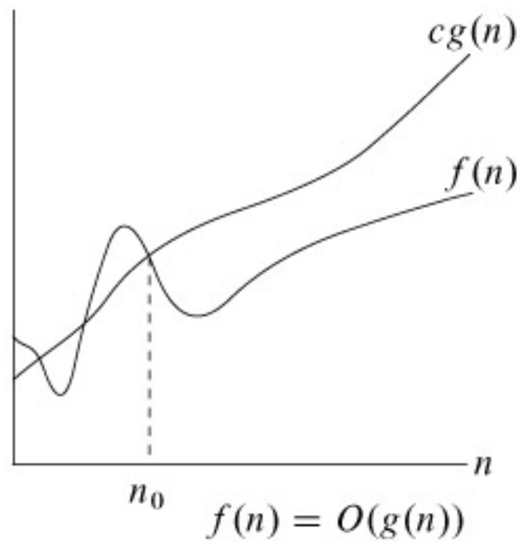
- **O(1): constant:** access an element of an array (of length n)
- **O(log n): logarithmic:** binary search through sorted array of length n
- **O(n): linear:** maximum element of list of length n
- **O(n log n): linearithmic:** mergesort a list of length n
- **$O(n^2)$: quadratic:** bubblesort an array of length n
- **$O(n^3)$: cubic:** matrix multiplication of n-by-n matrices
- **$O(2^n)$: exponential:** enumerate all integers of bit length n

...some of these are not obvious, require proof

Asymptotic bounds

Big Oh:

- *asymptotic upper bound*
- $O(g) = \{ f \mid \exists\, c>0, n_0>0 \,.\, \forall\, n \geq n_0 \,.\, f(n) \leq c\, g(n) \}$
- intuitions: $f \leq g$, f is at least as efficient as g

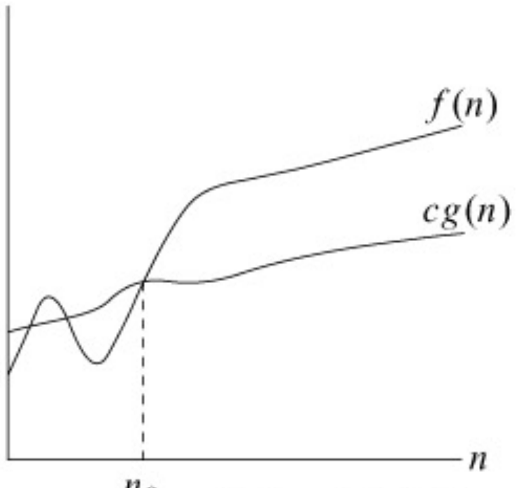


(b)

Asymptotic bounds

Big Omega

- *asymptotic lower bound*
- $\Omega(g) = \{ f \mid \exists\, c>0, n_0>0 \,.\, \forall\, n \geq n_0 \,.\, f(n) \geq c\, g(n) \}$
- intuitions: $f \geq g$, f is at most as efficient as g



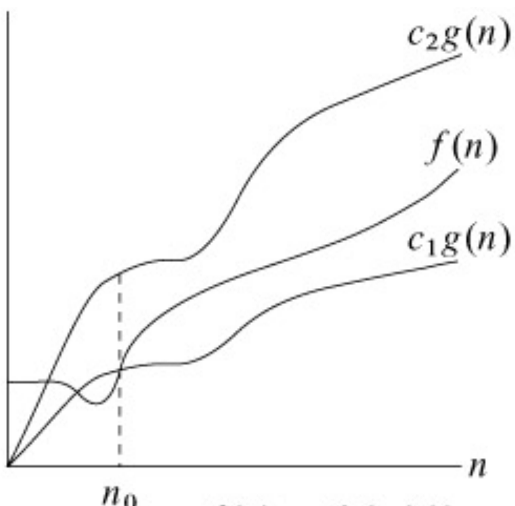
$f(n) = \Omega(g(n))$

(c)

Asymptotic bounds

Big Theta

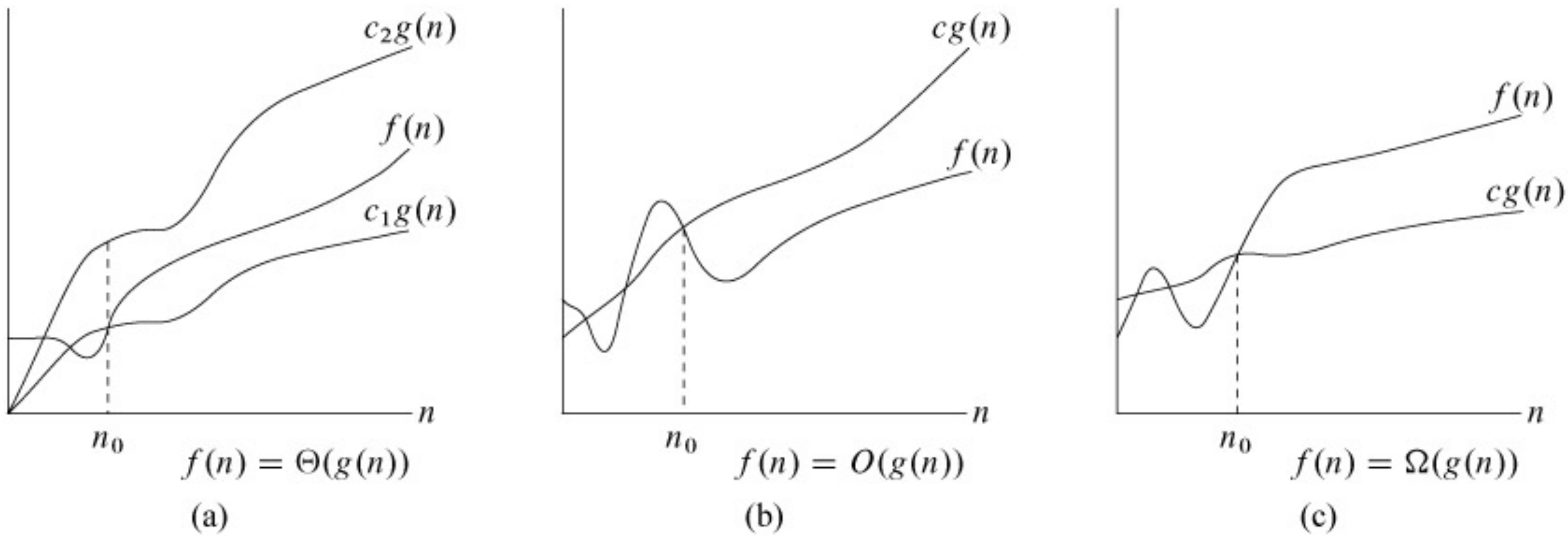
- *asymptotic tight bound*
- $\Theta(g) = O(g) \cap \Omega(g)$
- $\Theta(g) = \{ f \mid \exists\, c_1>0, c_2>0, n_0>0 \,.\, \forall\, n \geq n_0 \,.\, c_1\, g(n) \leq f(n) \leq c_2\, g(n) \}$
- intuitions: $f = g$, f is just as efficient as g
- beware: some authors write O(g) when they really mean Θ(g)



$f(n) = \Theta(g(n))$

(a)

Asymptotic bounds



[Cormen et al. *Introduction to Algorithms*, 3rd ed, 2009]

Alternative notions of efficiency

- Expected-case running time
  - Instead of worst case
  - Useful for randomized algorithms
  - Maybe less useful for deterministic algorithms
    - Unless you really do know something about probability distribution of inputs
    - All inputs are probably not equally likely
- Space
  - How much memory is used? Cache space? Disk space?
- Other resources
  - Power, network bandwidth, …

Upcoming events

- [this week] nothing

This is efficient.

THIS IS 3110