Proof 2

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In this document we present a proof of

\[
INDXS = \forall A : Type. \\
\forall x : A. \\
\forall lst : List(A). \\
\exists indxs : List(N). \\
\forall i : N. \ i \in_N indxs 
\iff i < ||lst|| \land x = lst[i] \in A
\]

that generates the following program:

```ocaml
let rec indexes' x lst n = 
    match lst with 
    | [] -> [] 
    | h :: t -> 
      let l = indexes' x t (n + 1) 
      in if x = h then n :: l else l ;;

let indexes x lst = indexes' x lst 0 ;;
```

First we move all our universally quantified variables to our context:

```plaintext
# N = unit
# A = unit
# x
# lst
```

Then, we prove the following property instead. Note that the \{n\ldots\} which is the set of natural numbers greater or equal to \(n\) is necessary to prove that this assertion is true.
We prove that our assertion implies our original goal. This corresponds to making the call `indexes’ x lst 0`.

As in our first proof, we prove this new formula by induction on L:
As in our first proof, the base case is trivial:

In the induction case, we first compute the recursive call `indexes’ x t (n + 1)`. This gives us a list `idxs` called 1 in `indexes’`'s definition.
As in our first proof, we check whether $x$ is equal to $u$, the head of our list $L$. We obtain two subgoals. In the first branch, we get to assume that $x$ and $u$ are equal and in the other branch we get to assume that they are different.
If $x$ is equal to $u$ then the list of indexes is $n :: \text{indxs}$. After doing some simple reasoning, we are left with two simple subgoals (see the two next figures).
1. \( x \equiv y \iff (\text{deq } x y) \text{A} \)
2. \( L : \text{A List} \)
3. \( u : \text{A} \)
4. \( v : \text{A List} \)
5. \( n : \text{Nat} \)
6. \( \text{idxs} : \text{C} \text{List} | (\forall i : \text{Nat}. \ ((i + n \in \text{idxs}) \iff (i < |v|) \land (v[i] = x))) \)
7. \( \text{t} : \text{deq } x u \)
8. \( \exists \text{idxs} : \text{C} \text{List} | (\forall i : \text{Nat}. \ ((i + n + 1 \in \text{idxs}) \iff (i < |v|) \land (v[i] = x)) \)
9. \( \text{BY} \) \( \text{InstCncl}[\text{"n / idxs"}] \) \( \text{THEN Auto} \) \( \text{THEN SmReasoner} \) \( \text{THEN Decide} \) \( i = 0 \) \( \text{THEN Auto} \)
10. \( i < (|v| + 1) \)
11. \( \text{BY} \)
12. \( i < (|v| + 1) \)
13. \( i \in \text{Nat} \)
14. \( (i + n = n) \lor (i + n \in \text{idxs}) \)
15. \( \text{BY} \)

* top 1 1 2 1 1 1
1. \( A, \text{deq}, x, L, \text{A} \)
2. \( u : \text{A} \)
3. \( v : \text{A List} \)
4. \( n : \text{Nat} \)
5. \( \text{idxs} : \text{C} \text{List} | (\forall i : \text{Nat}. \ ((i + n \in \text{idxs}) \iff (i < |v|) \land (v[i] = x))) \)
6. \( \text{t} : \text{deq } x u \)
7. \( \text{BY} \) \( \text{InstHyp}[\text{"i - 1"}] \) \( \text{THEN Auto} \) \( \text{THEN \text{Subst}} [\text{"i - 1"} + n + 1 - i + n] \) \( \text{THEN Auto} \) \( \text{THEN Rule } "-1" \)
8. \( \text{THEN Auto} \)
If \( x \) is not equal to \( u \) then the list of indexes is \( \text{indxs} \). After doing some simple reasoning, we are now left with three simple subgoals (see the two three figures). The first and third subgoals are trivial. Let us look at the second one.
# top 1 1 2 1 2

[+]A,deq,
[-]
3. \( \forall x, y : A. \ (x = y \iff \top(deq \times y)) \)\* 
4. \( x : A \)
5. \( L : A \) List
6. \( u : A \)
7. \( v : A \) List
8. \( \forall i : \mathbb{N}. \ (\langle i, n \rangle \in \text{indices}(\langle n \rangle) \iff (i < |v|) \land (v[i] = x)) \)
9. \( n : \mathbb{N} \)
10. \( \text{indices} : \langle n + 1 \rangle \) List
11. \( \forall i : \mathbb{N}. \ (i + n + 1 \in \text{indices}) \iff (i < |v|) \land (v[i] = x) \)
12. \( \neg \top(deq \times u) \)
[-]
\( \exists \text{indices}(\langle n \rangle) \) List \( \forall i : \mathbb{N}. \ (i + n \in \text{indices}) \iff (i < (|v| + 1)) \land ((u \div v)[i] = x) \)

```
# BY ['{LastConcl ['Indxs']}{HLN Auto}{HLN Spelvason}{HLN Decide 'i = 0'}{HLN Auto}]
```

# 1 1 2 1 2 1

[+]\* 
[+],
[-]
\( i < (|v| + 1) \)

```
# BY
```

# 1 1 2 1 2 2

[+],
[+],
[-]
\( u = x \)

```
# BY
```

# 1 1 2 1 2 3

[+],
[+],
[-]
\( i + n \in \text{indices} \)

```
# BY
```
In this subgoal we have to prove that $x$ is equal to $u$, assuming that $x$ is different from $u$. We can prove this, because it turns out that we can get a contradiction from our list of hypotheses. Hypothesis 14 says that $i + n$ is in the list $\text{indices}$. We also have that $i$ is 0. Therefore $n$ is in the list $\text{indices}$. This is not possible because $\text{indices}$ is the list of positions of $x$ in the tail of our list, which was obtained by applying $\text{indexes'}$ to $n + 1$, and $n$ must be greater or equal to $n + 1$ (see hypothesis 10).
10