CS 3110 — Data Structures and Functional Programming

Lecture 27
Fixpoints and Recursion

3 May 2012

Recursion in λ-calculus

Last time: encoded booleans and numbers in λ-calculus.

Can we use these to express the factorial function?

```
let rec fact n =
 if n=0 then 1 else n * fact (n-1)
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Yes... but we need a way to define recursive functions!

What about “Landin’s knot”?

```
let fact =
 let g : (int -> int) = ref (fun n -> 42) in
 let f n = if n=0 then 1 else n * !g (n-1) in
 g := f;
 fun n -> !g n
```

Won’t work—λ-calculus doesn’t have references!

Fixpoints

```
let t_fact g n = if n=0 then 1 else n * g (n-1)

let fact0 = (fun n -> 42) (* {} ok *)
let fact1 = t_fact fact0  (* {0} ok *)
let fact2 = t_fact fact1  (* {0,1} ok *)
let fact3 = t_fact fact2  (* {0,1,2} ok *)
          .
          .
          .
let fact = t_fact fact    (* {0,1,2,...} ok *)
```

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```

Definition (Fixpoint)

A *fixpoint* x of a function f satisfies $f(x) = x$.

So we want to find a fixpoint of `t_fact`.

Fixpoints in λ-calculus

Recall the λ-calculus term

$$omega \triangleq (\lambda x.\, x\, x)\ (\lambda x.\, x\, x)$$

which β-converts to itself in one step.

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If we interpose an arbitrary λ-calculus term F we get

$$\begin{array}{rl} & (\lambda x.\, F\,(x\, x))\ (\lambda x.\, F\,(x\, x)) \\ \Rightarrow & F\,((\lambda x.\, F\,(x\, x))\ (\lambda x.\, F\,(x\, x))) \end{array}$$

That is, a fixed point of F!

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That is, a fixed point of F!
The famous Y combinator is just

$$Y \triangleq \lambda f.(\lambda x.\, f\,(x\, x))\ (\lambda x.\, f\,(x\, x))$$

Factorial in λ-calculus

$$
\begin{aligned}
Y &\triangleq \lambda f.(\lambda x.\, f\,(x\,x))\;(\lambda x.\, f\,(x\,x))\\
t\_fact &\triangleq \lambda g.\,\lambda n.\, cond\;(iszero\;n)\;\overline{1}\;(mul\;n\;(g\;(pred n)))\\
fact &\triangleq Y\;t\_fact
\end{aligned}
$$

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Theorem (Correctness of *fact*)

$\forall n.\, fact\ \overline{n} = \overline{n!}$

Factorial in λ-calculus

$$Y \triangleq \lambda f.(\lambda x.\, f\,(x\,x))\,(\lambda x.\, f\,(x\,x))$$

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$$fact \triangleq Y\; t\_fact$$

Theorem (Correctness of *fact*)

$\forall n.\, fact\; \overline{n} = \overline{n!}$

Proof.

By induction on *n*... □

Review

Overview

- Functional Programming
- Data Structures
- Verification and Testing
- Concurrency
- Analysis of Algorithms
- Advanced Topics

Functional Programming

- OCaml Basics (syntax, evaluation)
- Types (tuples, records, variants, polymorphism)
- Higher-order functions (currying)
- Side-effects (printing, exceptions)
- Maps and folds (tail recursion)
- The Substitution Model

Functional Data Structures

- Basic Modules (signatures, structures)
- Basic data structures (stacks, queues, dictionaries)
- Advanced Modules (abstraction functions, representation invariants)
- Trees (red-black)
- Mutability (arrays, union-find, functional arrays)
- The Environment Model

Verification and Testing

- Logic (propositional, predicate)
- Induction
- Verification (total, partial correctness)

Concurrency

- Threads
- Locks and condition variables

Analysis of Algorithms

- Asymptotic complexity
- Recurrences and recursion trees
- Master method
- Substitution method
- Amortized analysis

Advanced Toics

- Memoization
- Locality and Memory Management
- Graph Algorithms
- Type Inference and Unification
- Laziness and Streams
- λ-calculus
- Fixpoints and Recursion

Thank you!