Lecture 27
Fixpoints and Recursion

3 May 2012
Recursion in $\lambda$-calculus

Last time: encoded booleans and numbers in $\lambda$-calculus. Can we use these to express the factorial function?

```ml
let rec fact n =
  if n=0 then 1 else n * fact (n-1)
```

Yes... but we need a way to define recursive functions! What about "Landin’s knot"?

```
let fact =
  let g : (int -> int) = ref (fun n -> 42) in
  let f n =
    if n=0 then 1 else n * !g (n-1) in
  g := f;
  fun n -> !g n
```

Won’t work—$\lambda$-calculus doesn’t have references!
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Fixpoints

```ocaml
let t_fact g n = if n=0 then 1 else n * g (n-1)

let fact0 = (fun n -> 42) (* {} ok *)
let fact1 = t_fact fact0 (* {0} ok *)
let fact2 = t_fact fact1 (* {0,1} ok *)
let fact3 = t_fact fact2 (* {0,1,2} ok *)

let fact = t_fact fact (* {0,1,2,...} ok *)
```
Fixpoints

\[
\text{let } \ t\_\text{fact} \ g \ n = \ \text{if } n=0 \ \text{then} \ 1 \ \text{else} \ n * g(n-1)
\]

\[
\begin{align*}
\text{let } \ \text{fact}0 &= (\text{fun } n \to 42) (* \ \{} \ ok \ *) \\
\text{let } \ \text{fact}1 &= \ t\_\text{fact} \ \text{fact}0 \ (* \ \{0\} \ ok \ *) \\
\text{let } \ \text{fact}2 &= \ t\_\text{fact} \ \text{fact}1 \ (* \ \{0,1\} \ ok \ *) \\
\text{let } \ \text{fact}3 &= \ t\_\text{fact} \ \text{fact}2 \ (* \ \{0,1,2\} \ ok \ *) \\
. \ \ \ \ . \ \ \ \ . \ \ \ \ . \ \ \ \ .
\text{let } \ \text{fact} &= \ t\_\text{fact} \ \text{fact} \ (* \ \{0,1,2,\ldots\} \ ok \ *)
\end{align*}
\]

**Definition (Fixpoint)**

A fixpoint \( x \) of a function \( f \) satisfies \( f(x) = x \).

So we want to find a fixpoint of \( t\_\text{fact} \).
Fixpoints in $\lambda$-calculus

Recall the $\lambda$-calculus term

$$\omega \triangleq (\lambda x. x x) (\lambda x. x x)$$

which $\beta$-converts to itself in one step.
Fixpoints in $\lambda$-calculus

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If we interpose an arbitrary $\lambda$-calculus term $F$ we get

$$ (\lambda x. F (x x)) (\lambda x. F (x x)) $$

$$ \Rightarrow F ((\lambda x. F (x x)) (\lambda x. F (x x))) $$

That is, a fixed point of $F$!
Fixpoints in $\lambda$-calculus

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That is, a fixed point of $F$!

The famous $Y$ combinator is just

$$Y \triangleq \lambda f.(\lambda x. f (x x)) (\lambda x. f (x x))$$
Factorial in $\lambda$-calculus

\[
Y \triangleq \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))
\]

\[
t_{\text{fact}} \triangleq \lambda g. \lambda n. \text{cond} (\text{iszero } n) \overline{1} (\text{mul } n (g (\text{predn})))
\]

\[
\text{fact} \triangleq Y t_{\text{fact}}
\]
Factorial in $\lambda$-calculus

$$Y \triangleq \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x))$$

$$t_{\text{fact}} \triangleq \lambda g. \lambda n. \text{cond (iszero n) } \overline{1} (\text{mul n } (g \ (\text{predn})))$$

$$\text{fact} \triangleq Y \ t_{\text{fact}}$$

Theorem (Correctness of fact)

$$\forall n. \text{fact } \overline{n} = \overline{n}!$$
Factorial in $\lambda$-calculus

\[ Y \triangleq \lambda f. (\lambda x. f (x x)) (\lambda x. f (x x)) \]
\[ t_{fact} \triangleq \lambda g. \lambda n. \text{cond (iszero } n) \ 1 \ (\text{mul } n \ (g \ (\text{predn}))) \]
\[ \text{fact} \triangleq Y \ t_{fact} \]

Theorem (Correctness of fact)

\[ \forall n. \text{fact } \overline{n} = \overline{n}! \]

Proof.

By induction on $n$...
Review
Overview

- Functional Programming
- Data Structures
- Verification and Testing
- Concurrency
- Analysis of Algorithms
- Advanced Topics
Functional Programming

- OCaml Basics (syntax, evaluation)
- Types (tuples, records, variants, polymorphism)
- Higher-order functions (currying)
- Side-effects (printing, exceptions)
- Maps and folds (tail recursion)
- The Substitution Model
Functional Data Structures

- Basic Modules (signatures, structures)
- Basic data structures (stacks, queues, dictionaries)
- Advanced Modules (abstraction functions, representation invariants)
- Trees (red-black)
- Mutability (arrays, union-find, functional arrays)
- The Environment Model
Verification and Testing

- Logic (propositional, predicate)
- Induction
- Verification (total, partial correctness)
Concurrency

- Threads
- Locks and condition variables
Analysis of Algorithms

- Asymptotic complexity
- Recurrences and recursion trees
- Master method
- Substitution method
- Amortized analysis
Advanced Topics

- Memoization
- Locality and Memory Management
- Graph Algorithms
- Type Inference and Unification
- Laziness and Streams
- $\lambda$-calculus
- Fixpoints and Recursion
Thank you!