Discrete Structures *CS2800*	*Prelim 1*— ***Solutions***

1. *[5 points] Recall that $[X \to Y]$ denotes the set of all (total) functions from X to Y, and Y^n denotes $Y \times Y \times \cdots \times Y$ (n times).*

   Let $X = \{x_1, x_2, \dots, x_n\}$ be a finite set with cardinality n. Define a bijection from $[X \to Y]$ to Y^n.

   Solution Let $F : [X \to Y] \to Y^n$ be given by $F(f) = (f(x_1), f(x_2), \dots, f(x_n))$. This function is injective: if $F(f) = F(g)$ then $f(x_1) = g(x_1)$, $f(x_2) = g(x_2)$, and in fact $f(x_i) = g(x_i)$ for all $i \leq n$. Thus $f = g$.

   This function is surjective: given a tuple $t = (y_1, y_2, \dots, y_n)$, let $f : X \to Y$ be given by $f(x_i) = y_i$. Then clearly $F(f) = t$. Thus F is bijective.

   A few comments on the proof: some poeople observed that $|X \to Y| = |Y^n| = |Y|^n$, and thus there must be a bijection. This is true (and typically got 2/5 if there were no other problems) but the question asked you to *define* a bijection. If you just say that there has to be a bijection, that isn't good enough; you have to say what it is.

2. *[5 points] Let $\{1, 2, 3\}^\omega$ be the set of infinite sequences containing only the numbers 1, 2, and 3. For example, some sequences of this kind are:*

$$(1, 1, 1, 1, 1, 1, \dots)$$
$$(2, 2, 2, 2, 2, 2, \dots)$$
$$(3, 2, 1, 3, 2, 1, \dots)$$

   Prove that $\{1, 2, 3\}^\omega$ is uncountable.

   Solution Suppose (for the sake of contradiction) that $X = \{1, 2, 3\}^\omega$ is countable. Then there exists a surjection $f : \mathbb{N} \to X$. We will show that f is *not* a surjection by constructing a sequence s_D that is not in the image of f.

   Form the ith element of the sequence s_D by adding one to the ith element of $f(i)$ (wrapping around to 1 if $f(i)_i = 3$). The $s_D \neq f(i)$ for any i, because it differs in the ith place. Therefore s_D is not in the image of f, contradicting the assumption that f is surjective. Thus X is not countable.

3. *[4 points] Prove by induction that*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

   for all $n \geq 1$.

   Solution Let $P(n)$ be the statement

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

   Base case: $P(1)$ is true, because $1 \cdot 2 = 1 \cdot 2 \cdot 3/3$.

   Inductive step: Assume $P(n)$. We wish to show $P(n+1)$, i.e.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n+1)(n+2) = \frac{(n+1)(n+2)(n+3)}{3}$$

Well,

$$\begin{aligned}
1\cdot 2+2\cdot 3+3\cdot 4+\cdots+(n+1)(n+2) &= (1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1))+(n+1)(n+2) \\
&= \frac{n(n+1)(n+2)}{3}+(n+1)(n+2) && \text{by } P(n) \\
&= \frac{n(n+1)(n+2)+3(n+1)(n+2)}{3} && \text{common denominat} \\
&= \frac{(n+3)(n+1)(n+2)}{3} && \text{grouping } (n+1)(n \text{ -}
\end{aligned}$$

as required.

To get full credit, you needed to have a clearly-stated induction hypothesis $P(n)$. You lost 1 point for the standard errors that we also deducted one point for on the homework: e.g., including the "for all n" inside the $P(n)$ or treating $P(n)$ as a number. ($P(n)$ is *not* a number! It's an English statement that has an n in it. You can't write, for example, "$P(n) = n(n+1)$". That doesn't make sense.)

4. *[6 points: 2+4] (a) Explain carefully what the flaw is with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.*

   We use regular induction to prove that $P(n)$ holds for all $n \geq 3$, where $P(n)$ says that postage of n cents cents can be formed using just three-cent and four-cent stamps.

   Base Case: *We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.*

   Inductive Step: *Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k+1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four- cent stamps by three three-cent stamps.*

   (b) Show that every postage of six cents or more can be formed using just three-cent and four-cent stamps.

   Solution (a) In doing the inductive step, you don't know that a three-cent stamp or two four-cent stamps were used in paying for k cents. In particular, if $k = 4$, only one four-cent stamp is used. So the argument going from 4 to 5 fails. (We also reluctantly accepted as an error in the proof that the inductive step should have said "Assume that we can form postage of j cents for all nonnegative integers j with $3 \leq j \leq k$"; that is, we mut start at 3 (because that's the base case), not 0 (which is what we're doing if we just say "for all nonnegative integers j with $j \leq k$".)

   (b) Use strong induction to prove $P(n)$ (where $P(n)$ is as above) for $n \geq 6$.

   Base Case: We can form postage of six cents using two three-cent stamps.

   Inductive Step: Suppose that $P(j)$ holds for all $j \leq n$. We want to prove $P(n+1)$.

   If $n+1 = 7$, we can use a three- and a four-cent stamp; if $n+1 = 8$, we can use two four-cent stamps. (It's OK to make 7 and 8 part of the base case.) If $n+1 \geq 9$, then $n \geq 8$, so $n+1-3 = n-2 \geq 6$. That means that, using the induction hypothesis, we can pay for $n-2$ cents of postage using three- and four-cent stamps. Add one more three-cent stamp. That will pay for $n+1$ cents of postage.

   As in question 3, we deducted points for including "for all n" in the statement $P(n)$ or somehow treating $P(n)$ as a number.

5. *[7 points: 4+3] (a) Six women and nine men are on the faculty of a schools CS department. The individuals are distinguishable. How many ways are there to select a committee of 5 members if at least 1 woman must be on the committee?*

   (b) How many ways can Mr. and Mrs. Sweettooth distribute 13 identical pieces of candy to their three children so that each child gets at least one piece of candy?