Discrete Structures	*Prelim 1— **Solutions***
CS2800	*Selected problems from past exams*

1. *True/false. For each of the following statements, indicate whether the statement is true or false. Give a one or two sentence explanation for your answer.*

   (a) *A proof that starts "Choose an arbitrary $y \in \mathbb{N}$, and let $x = y^2$" is likely to be a proof that $\forall y \in \mathbb{N}, \forall x \in \mathbb{N}, \ldots$.*

   Solution **False.** This would only be a proof that $\exists x \in \mathbb{N}$ with some property, not a proof that $\forall x \in \mathbb{N}$ the property holds.

   (b) *The set of real numbers ($\mathbb{R}$) is countable.*

   Solution **False.** We proved this in class using diagonalization.

   (c) *The set of rational numbers ($\mathbb{Q}$) is countable.*

   Solution **True.** We proved this in class by giving a procedure for listing all of the rational numbers (by putting them in a table and traversing the diagonals of the table).

   (d) *The sentence "everybody can fool Mike" is false if and only if the sentence "nobody can fool Mike" is true.*

   Solution **False.** The negation would be "somebody can't fool Mike".

   (e) *Recall that $[X \to Y]$ denotes the set of functions with domain X and codomain Y. Let $f : 2^S \to [S \to \{0,1\}]$ be given by $f(X) ::= h$ where $h : S \to \{0,1\}$ is given by $h(s) ::= 0$. f is injective.*

   Solution **False.** f always returns the same thing, so it can't be one to one. For example, choose any two different subsets X_1 and X_2 of S; then $f(X_1) = h = f(X_2)$.

   (f) *f as just defined is surjective.*

   Solution **False.** Choosee any function $h' : S \to \{0,1\}$ other than h. Since f only outputs h, it never outputs h'.

2. *Prove the following claim using induction: for any $n \geq 0$, $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$*

Solution Base case: when $n = 0$, the left hand side is $2^0 = 1$ and the right hand side is $2^2 - 1 = 1$, and they are clearly the same.

Inductive step: Choose an arbitrary n and assume that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$ (this is the inductive hypothesis).

We wish to show that $\sum_{i=0}^{n+1} 2^i = 2^{n+2} - 1$. We compute:

$$\begin{aligned}
\sum_{i=0}^{n+1} 2^i &= \sum_{i=0}^{n} 2^i + 2^{n+1} && \text{arithmetic} \\
&= (2^{n+1} - 1) + 2^{n+1} && \text{by the inductive hypothesis} \\
&= 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1
\end{aligned}$$

as required.

3. *Complete the following diagonalization proof:*

   Claim: *$X = [\mathbb{N} \to \mathbb{N}]$ is uncountable.*

   Proof: *We prove this claim by contradiction. Assume that X is countable. Then there exists a function F :* ***FILL IN*** *that is* ***FILL IN****.*

   Write $f_0 = F(0)$, $f_1 = F(1)$, and so on. We can write the elements of X in a table:

	0	*1*	*2*	$\cdots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\ddots$

   Let f_D : ***FILL IN*** *be given by $f_D : x \mapsto$* ***FILL IN***

   Then ***FILL IN***

   This is a contradiction because ***FILL IN****.*

   Solution **Claim:** $X = [\mathbb{N} \to \mathbb{N}]$ is uncountable.

   Proof: We prove this claim by contradiction. Assume that X is countable. Then there exists a function $F : \mathbb{N} \to X$ that is **surjective**.

   Write $f_0 = F(0)$, $f_1 = F(1)$, and so on. We can write the elements of X in a table:

	0	1	2	$\cdots$
f_0	$f_0(0)$	$f_0(1)$	$f_0(2)$	$\cdots$
f_1	$f_1(0)$	$f_1(1)$	$f_1(2)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$		$\ddots$

   Let $f_D : \mathbb{N} \to \mathbb{N}$ be given by $f_D : x \mapsto 1 + f_x(x)$

   Then f_D **is not in the table, because for any i, it differs from f_i on input i.**

   This is a contradiction because **we assumed F was surjective**.

4. *Suppose you are given a function $f : \mathbb{N} \to \mathbb{N}$, and are told that $f(1) = 1$ and for all n, $f(n) \leq 2f(\lfloor n/2 \rfloor) + 1$.*

   Use strong induction on n to prove that for all $n \geq 2$, $f(n) \leq 2n \log_2 n$.

   You may write $\log$ *to indicate* $\log_2$*. Here is a reminder of some facts about $\lfloor x \rfloor$ and $\log x$:*

   - $\lfloor x \rfloor \leq x$
   - $\log 1 = 0$, $\log 2 = 1$
   - $\log(x/2) = \log x - 1$
   - $\log(2^x) = x$
   - $\log(x^2) = 2 \log x$
   - *if $x \leq y$ then* $\log x \leq \log y$

Solution In the base case, we need to show $f(2) \leq 4\log 2 = 2$. But we are given that $f(2) \leq 2f(1)+1 = 3 \leq 4$, as required.

For the inductive step, choose $n > 2$, and assume that for all $k < n$, $f(k) \leq 2k\log k$. We must show that $f(n) \leq 2n\log n$.

We compute:

$$
\begin{array}{rll}
f(n) & \leq 2f(\lfloor n/2 \rfloor) + 1 & \text{given} \\
& \leq 4\lfloor n/2 \rfloor \log\lfloor n/2 \rfloor + 1 & \text{by inductive hypothesis} \\
& \leq 4(n/2)\log\lfloor(\rfloor n/2) + 1 & \text{by facts stated in question} \\
& \leq 2n\log n - 2n + 1 \leq 2n\log n + (1-2n) & \text{arithmetic} \\
& \leq 2n\log n + 0 & \text{since } n > 2 \text{ so } 1 - 2n < 0 \\
& = 2n\log n & \text{as required.}
\end{array}
$$

5. *Which of the following sets are countably infinite and which are not countably infinite? Give a one to five sentence justification for your answer.*

   (a) *The set Σ^* containing all finite length strings of 0's and 1's.*

   Solution This set is countable. You can list all strings of length 0, then all strings of length one, then all strings of length 2, and so on.

   (b) *The set $2^{\mathbb{N}}$ containing all sets of natural numbers.*

   Solution This set is not countable. If it were, we could put all of the sets in a table:

	0	1	2	$\cdots$
S_1	$0 \in S_1$	$\notin$	$\in$	$\cdots$
S_2	$\notin$	$\notin$	$\notin$	$\cdots$
S_3	$\in$	$\notin$	$\in$	$\cdots$

   We can then construct the set S_D by swapping everything on the diagonal ($S_D = \{i \mid i \notin S_i\}$). Then $S_D \neq S_k$ for any k, because $k \in S_D$ if and only if $k \notin S_k$. Thus S_D is not in the table, which contradicts the fact that the table contained all sets.

   (c) *The set $\mathbb{N} \times \mathbb{N}$ containing all pairs of natural numbers.*

   Solution This set is countable. You can put all of the pairs in a table, and then map the natural numbers to the pairs by tracing diagonals of the table.

   (d) *The set $[\mathbb{N} \to \{0,1\}]$ containing all functions from $\mathbb{N}$ to $\{0,1\}$.*

   Solution This set is not countable. There is a bijection between $[\mathbb{N} \to \{0,1\}]$ and $2^{\mathbb{N}}$, and we showed above that $2^{\mathbb{N}}$ is uncountable. Alternatively, you can diagonalize directly using the function $f : n \mapsto f_n(n) + 1$ or similar.

   Be sure to include enough detail:

   - *If listing elements, be sure to clearly state how you are listing them;*
   - *If diagonalizing, be sure it is clear what your diagonal construction is;*

- *If providing a function, make sure it is clear what the output is on a given input.*

6. *For any function $f : A \to B$ and a set $C \subseteq A$, define $f(C) = \{f(x) \mid x \in C\}$. That is, $f(C)$ is the set of images of elements of C. Prove that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ for all $C_1, C_2 \subseteq A$.*

   (Hint: *one way to prove this is from the definition of set equality: $A = B$ iff $A \subseteq B$ and $B \subseteq A$.)*

7. *The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively as follows:*

$$\begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \quad \textit{for } n \geq 2 \end{aligned}$$

   That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove by induction that for all natural numbers n (including 0):

$$\sum_{i=0}^{n} F_i \;=\; F_{n+2} - 1$$

8. *Prove by induction that for any integer $n \geq 3$, $n^2 - 7n + 12$ is non-negative.*

9. (a) *Write the definition of "$f : A \to B$ is injective" using formal notation ($\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $=$, $\neq$, $\ldots$).*

   (b) *Similarly, write down the definition of "$f : A \to B$ is surjective".*

   (c) *Write down the definition of "A is countable". You may write "f is surjective" or "f is injective" in your expression.*

10. *Recall that the composition of two functions $f : B \to C$ and $g : A \to B$ is the function $f \circ g : A \to C$ defined as $(f \circ g)(x) = f(g(x))$. Prove that if f and g are both injective, then $f \circ g$ is injective.*

11. *For each of the following functions, indicate whether the function f is injective, whether it is surjective, and whether it is bijective. Give a one sentence explanation for each answer.*

    (a) $f : \mathbb{N} \to \mathbb{N}$ *given by* $f : x \to x^2$

    (b) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f : x \to x^2$

    (c) $f : X \to [Y \to X]$ *given by* $f : x \mapsto h_x$ *where* $h_x : Y \to X$ *is given by* $h_x : y \mapsto x$.

12. *Recall that the Fibonacci numbers $F_1, F_2, F_3, \ldots$ are defined inductively as follows:*

$$\begin{aligned} F_1 &= 1 \\ F_2 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \quad \textit{for } n \geq 3 \end{aligned}$$

    That is, each Fibonacci number is the sum of the previous two numbers in the sequence. Prove that:

$$\sum_{i=1}^{n} F_i^2 \;=\; F_n F_{n+1}$$

13. *A chocolate bar consists of n identical square pieces arranged in an unbroken rectangular grid. For instance, a 12-piece bar might be a 3×4, 2×6 or 1×12 grid. A single snap breaks the bar along a straight line separating the squares, into two smaller rectangular pieces. Prove that regardless of the initial dimensions of the bar, any n-piece bar requires exactly $n - 1$ snaps to break it up into individual squares.*

14. *Briefly and clearly identify the errors in each of the following proofs:*

(a) **Proof that 1 is the largest natural number:** *Let n be the largest natural number. Then n^2, being a natural number, is less than or equal to n. Therefore $n^2 - n = n(n-1) \leq 0$. Hence $0 \leq n \leq 1$. Therefore $n = 1$.*

(b) **Proof that $2 = 1$:** *Let $a = b$.*

$$\begin{aligned} \Rightarrow \quad & a^2 = ab \\ \Rightarrow \quad & a^2 - b^2 = ab - b^2 \\ \Rightarrow \quad & (a+b)(a-b) = b(a-b) \\ \Rightarrow \quad & a + b = b \end{aligned}$$

Setting $a = b = 1$, we get 2 = 1.

(c) **Proof that $(a+b)(a-b) = a^2 - b^2$:**

$$\begin{aligned} \textit{To prove:} \quad & (a+b)(a-b) = a^2 - b^2 \\ \Rightarrow \quad & a^2 - ab + ab - b^2 = a^2 - b^2 \\ \Rightarrow \quad & a^2 - b^2 = a^2 - b^2 \end{aligned}$$

. . . which is true, hence the result is proved.

15. *Prove that $7^m - 1$ is divisible by 6 for all positive integers m.*

16. *Prove that*

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

for all positive integers n.

17. *Prove by induction that the sum of the interior angles of a convex*[1] *polygon with n sides (and hence n vertices) is $180(n-2)$ degrees. You may use the fact that the sum of the interior angles of a triangle is 180 degrees. You do not need to prove straightforward geometrical facts rigorously (check with us if unsure).*

18. *In a permutation of the set $\{1, 2, \ldots, n\}$, a pair i, j is* out of order *if $i < j$ but i occurs after j in the permutation. In a random permutation of the set $\{1, 2, \ldots, n\}$ with all permutations equally likely, what is the expected number of pairs that are out of order?*

Solution $\binom{n}{2}/2$

19. *Consider the following assertion.*

Assuming everyone has three initials, there are at least 6 people in California (population 38 million) with the same initials and the same birthday (the same day of the year, but not necessarily the same year).

Describe a simple calculation you could do to verify this assertion.

Solution Using the extended pigeonhole principle, check whether $\lceil \frac{38{,}000{,}000}{366 \cdot 26^3} \rceil \geq 6$.

20. *Give an expression describing the number of different ways the following things can happen. No credit will be given for just the value, even if correct.*

[1] A polygon is convex if, for all vertices p and q of the polygon, the line joining p and q lies entirely within the polygon.

(a) *During your pregnancy, you decided on a list of 23 girls' first names and 16 boys' first names, as well as a list of 11 gender-neutral middle names. To your surprise, you had quintuplets, two boys and three girls. Now you must select a first and a middle name for each child from the lists. The names must all be different.*

Solution $P(16,2) \cdot P(23,3) \cdot P(11,5)$ or equivalent, e.g. $(16!/14!)(23!/20!)(11!/6!)$, $16 \cdot 15 \cdot 23 \cdot 22 \cdot 21 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

(b) *A professor teaching discrete structures is making up a final exam. He has a stash of 24 questions on probability, 16 questions on combinatorics, and 10 questions on logic. He wishes to put five questions on each topic on the exam.*

Solution $\binom{24}{5}\binom{16}{5}\binom{10}{5}$ or equivalent

(c) *The very same professor wants to assign points to the 15 problems so that each problem is worth at least 5 points and the total number of points is 100.*

Solution $\binom{25+15-1}{25}$ or $\binom{25+15-1}{15-1}$ or equivalent

(d) *There are 30 graders to grade the final exam, and the professor would like to assign two graders to each of the 15 problems.*

Solution $\binom{30}{2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2\ 2}$ or $\frac{30!}{2^{15}}$ or $\binom{30}{2}\binom{28}{2}\binom{26}{2}\cdots\binom{2}{2}$

21. *Give an expression describing the number of different ways the following things can happen. Your expression may involve binomial coefficients, multinomial coefficients, Stirling numbers of the second kind, factorial expressions, r-permutations, or whatever else you need, but do not evaluate the expression. No credit will be given for just the value, even if correct.*

(a) *You must choose a password consisting of 6, 7, or 8 letters from the 26-letter English alphabet* $\{a, b, \ldots, z\}$.

Solution $26^6 + 26^7 + 26^8$

(b) *In a poker game, you are dealt a* full house*, a five-card hand containing three of a kind and a pair of another kind; for example, three kings and two sixes.*

Solution $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$

(c) *Your team is in the championship game of a soccer tournament. The score is tied at full time and the winner will be decided by penalty kicks. As coach, you must choose a sequence of five different players out of 11 to take the kicks.*

Solution $P(11, 5) = 11!/(11-5)! = 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7$

(d) *You have ten rings, all with different gemstones. You wish to bequeath them to your five children so that each child inherits two of the rings.*

Solution $\binom{10}{2\ 2\ 2\ 2\ 2} = \dfrac{10!}{2!\,2!\,2!\,2!\,2!}$

(e) *You have \$400 to donate to charity, which you would like to distribute among your five favorite charities so that each receives an integral number of dollars.*

Solution $\binom{400+5-1}{400}$

(f) *The same as (e), but you also wish to ensure that each of the five charities receives at least \$20.*

Solution $\binom{300+5-1}{300}$