Discrete Structures	Prelim 1 sample questions
CS2800	Selected problems from past exams and a few additional problems

1. Compute $(10101)_2 + (101)_2$. Express your answer in both binary and decimal.

2. In this problem, we are working mod 7, i.e. $\equiv$ denotes congruence mod 7 and $[a]$ is the equivalence of a mod 7.
   (a) What are the units of $\mathbb{Z}_7$? What are their inverses?
   (b) Compute $[2]^{393}$.

3. Use Euler's theorem and repeated squaring to efficiently compute $8^n \mod 15$ for $n = 5$, $n = 81$ and $n = 16023$. Hint: you can solve this problem with 4 multiplications of single digit numbers. Please fully evaluate all expressions for this question (e.g. write 15 instead of $3 \cdot 5$).

4. (a) Recall Bézout's identity from the homework: for any integers n and m, there exist integers s and t such that $gcd(n, m) = sn + tm$. Use this to show that if $gcd(k, m) = 1$ then $[k]$ is a unit of $\mathbb{Z}_m$.
   (b) Use part (a) to show that if p is prime, then $\phi(p) = p - 1$.
   (c) Use Euler's theorem to compute $3^{38} \mod 37$ (note: 37 is prime).

5. (a) What are the units of $\mathbb{Z}$ mod 12?
   (b) What are their inverses?
   (c) What is $\phi(12)$?

6. Suppose we pick a bit string of length 4 at random, all bit strings equally likely. Consider the following events:
   E_1: the string begins with 1.
   E_2: the string ends with 1.
   E_3: the string has exactly two 1's.
   (a) Find $\Pr(E_1)$, $\Pr(E_2)$, and $\Pr(E_3)$.
   (b) Find $\Pr(E_1 \mid E_3)$.
   (c) Find $\Pr(E_2 \mid E_1 \cap E_3)$.
   (d) Are E_1 and E_2 independent? Justify your answer.
   (e) Are E_2 and E_3 independent? Justify your answer.

7. Give the *formal mathematical definition* of each of the following terms. Your definitions should be valid for a finite or countably infinite sample space S.
   (a) Probability distribution on S.
   (b) Event.
   (c) Probability of an event E, given a probability distribution Pr on S.
   (d) Conditional probability of E given F.

(e) Real-valued random variable X on S.

(f) Expectation of a random variable X.

(g) Variance of a random variable X.

8. Give an expression describing the probability of the following events. Evaluate the expression if it is easy to do so.

   (a) A fair coin is flipped 100 times giving exactly 50 heads.

   (b) A fair coin is flipped 100 times giving at most 50 heads.

   (c) A roll of two fair dice yields a sum of 7.

   (d) The conditional probability that the last card dealt in a 5-card poker hand yields a straight (five cards in sequence, irrespective of suit) given that the first four cards dealt were $5\spadesuit$, $6\heartsuit$, $7\diamondsuit$, $8\clubsuit$.

   (e) A controversial bill before the Senate is supported by 50 of 55 Democrats who will vote for the bill and opposed by 40 of 45 Republicans who will vote against the bill. Of the remaining 10 undecided Senators, each Democrat will vote in favor with probability 1/2 and each Republican against with probability 9/10, independent of the other votes. The bill requires 51 votes to pass. What is the probability that it passes?

9. The following questions refer to successive rolls of a fair six-sided die, where each roll yields an integer m in the range $1 \leq m \leq 6$ independently with uniform probability.

   (a) What is the expected number of times that 2 is rolled in 12 rolls?

   (b) What is the expected value of each roll?

   (c) What is the expected sum of four rolls?

   (d) What is the expected number of rolls before seeing the first 6?

10. The following questions refer to three independent flips of a fair coin.

   (a) Give an example of three events that are mutually independent.

   (b) Give an example of three events that are pairwise independent but not mutually independent.

   (c) Give an example of three events, no pair of which are independent.

11. Give an expression describing the probability of the following events. You do not need to evaluate the expression.

   (a) You obtain at least three consecutive heads in four flips of a fair coin.

   (b) In a poker game, you are dealt a five-card hand containing four aces.

   (c) The conditional probability that the first flip of two flips of a fair coin comes up heads, given that at least one of them does.

   (d) A 13-card bridge hand contains all cards of the same suit.

   (e) A randomly chosen number between 0 and 99 (inclusive) is divisible by 4.

12. Let $S = \{a, b\}$, and the function P given by the following table:

A	$P(A)$
$\emptyset$	0
$\{a\}$	1/4
$\{b\}$	3/4
$\{a, b\}$	1

   (a) **(1 point)** What are the events of S?

   (b) **(3 points)** Prove that (S, P) is a probability space.

   (c) **(1 point)** Are $\{a\}$ and $\{b\}$ independent? Explain.

13. **(2 points)** In an infamous criminal case, a mother was accused of murdering her two infant sons. A well-known statistician testified that the chance that *both* deaths were natural was infinitesimal. He proposed the following calculation:

$$P(D_1 \cap D_2 \mid I) = P(D_1 \mid I) \cdot P(D_2 \mid I)$$

   where D_1 and D_2 are the events that the two children respectively died, and I is the event that the mother is innocent.

   Since natural infant death is rare in the family's demographic, both probabilities on the right hand side are tiny: about 1/8543. Plugging in the values, we obtain $P(D_1 \cap D_2 \mid I) \approx 1/73{,}000{,}000$. Based on this, the mother was found guilty and imprisoned.

   Four years later, the ruling was overturned on grounds of faulty statistics. There are **two** significant errors in the reasoning above. Briefly and clearly identify both.

14. **(3 points)** Let E be a herd of 100 elephants. The herd contains 10 adult males, 60 adult females and 30 babies. It is known[1] that the adult elephants have an average surface area of 17m^2, and the babies have an average surface area of 4m^2. A biologist, unaware of these statistics, picks an elephant uniformly at random from E and measures its surface area (after temporarily and painlessly tranquilizing it). If the measured surface area is represented as a random variable, what are its (a) domain, (b) codomain, and (c) expectation (show your calculations)?

   (*Note:* There are many correct answers for (a) and (b). Pick any one.)

   (a) Domain: E, or $\{\text{adult}, \text{baby}\}$, or $\{\text{adult male}, \text{adult female}, \text{baby}\}$, or other reasonable variation.

   (b) Codomain: $\mathbb{R}$, or $\mathbb{R}^+$, or $\{17, 4\}$, or $\{17\text{m}^2, 4\text{m}^2\}$, or other reasonable variation.

[1] K. P. Sreekumar and G. Nirmalan, "Estimation of the total surface area in Indian elephants (Elephas maximus indicus)", Veterinary Research Communications, 1990;14(1):5-17.

(c) Expectation: the numerical answer (in m^2) is $17 \times 0.7 + 4 \times 0.3 = 13.1$. There are many ways to calculate this, depending on how you choose your domain. E.g. let's choose the domain E, call the random variable A, and for notational convenience (without loss of generality) assume the adult elephants are numbered 1 to 70, and the young 'uns 71-100. Then you might write:

$$\begin{aligned}
\text{Expectation of } A &= \sum_{i=1}^{100} A(elephant_i)P(elephant_i) \\
&= \sum_{i=1}^{70} A_{adult}P(elephant_i) + \sum_{i=71}^{100} A_{baby}P(elephant_i) \\
&= \sum_{i=1}^{70} 17 \cdot \frac{1}{100} + \sum_{i=71}^{100} 4 \cdot \frac{1}{100} \\
&= 17 \cdot \frac{70}{100} + 4 \cdot \frac{30}{100}
\end{aligned}$$

15. There are two cards: one is red on both sides and the other is black on one side and red on the other. One of the two cards is chosen at random and a side is picked at random and shown to you..
    (a) Draw a probability tree to describe the situation.
    (b) What is the sample space, and what is hte probability of each outcome.
    (c) Suppose that the side shown to you is red. What is the probabiity that the other side of the card is red?
16. **(0 points)** How would you (humanely) measure the surface area of an elephant?