Functions and Inverses

CS 2800: Discrete Structures, Spring 2015

Sid Chaudhuri
Recap: Relations and Functions

- A **relation** between sets $A$ (the **domain**) and $B$ (the **codomain**) is a set of ordered pairs $(a, b)$ such that $a \in A$, $b \in B$ (i.e. it is a subset of $A \times B$)
  - The relation maps each $a$ to the corresponding $b$
    - Neither all possible $a$'s, nor all possible $b$'s, need be covered
    - Can be one-one, one-many, many-one, many-many

![Diagram of relations and functions]

- Cartesian product
Recap: Relations and Functions

- A **function** is a relation that maps *each* element of $A$ to a *single* element of $B$
  - Can be one-one or many-one
  - All elements of $A$ must be covered, though not necessarily all elements of $B$
  - Subset of $B$ covered by the function is its *range/image*
Recap: Relations and Functions

- Instead of writing the function $f$ as a set of pairs, we usually specify its domain and codomain as:

  $$f : A \rightarrow B$$

  ... and the mapping via a rule such as:

  $$f(Heads) = 0.5, \quad f(Tails) = 0.5$$

  or $$f : x \mapsto x^2$$

The function $f$ maps $x$ to $x^2$
Recap: Relations and Functions

- Instead of writing the function $f$ as a set of pairs, we usually specify its domain and codomain as:
  
  \[ f : A \rightarrow B \]

  ... and the mapping via a rule such as:
  
  \[ f(\text{Heads}) = 0.5, \quad f(\text{Tails}) = 0.5 \]

  or  \[ f : x \mapsto x^2 \]

- **Note:** the function is $f$, not $f(x)$!
  - $f(x)$ is the value assigned by the function $f$ to input $x$
Recap: Injectivity

- A function is **injective** (one-to-one) if every element in the domain has a unique image in the codomain
  - That is, \( f(x) = f(y) \) implies \( x = y \)
Recap: Surjectivity

- A function is **surjective (onto)** if every element of the codomain has a preimage in the domain.
  - That is, for every $b \in B$ there is some $a \in A$ such that $f(a) = b$.
  - That is, the codomain is equal to the range/image.
Recap: Surjectivity

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- That is, for every $b \in B$ there is some $a \in A$ such that $f(a) = b$.
- That is, the codomain is equal to the range/image.
Recap: Bijectivity

- A function is **bijective** if it is both surjective and injective
Composition of Functions

- The **composition** of two functions
  \[ f : B \to C \]
  \[ g : A \to B \]
  is the function \( f \circ g : A \to C \) defined as
  \[ f \circ g : x \mapsto f( g(x) ) \]
Composition of Functions

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  \[ f : B \rightarrow C \]
  
  \[ g : A \rightarrow B \]
  
  is the function \( f \circ g : A \rightarrow C \) defined as
  
  \[ f \circ g : x \mapsto f( g( x ) ) \]
  
  \[ g \circ f \text{ is not possible unless } A = C ! \]
Factoid of the Day #1

Composition is \textit{associative}

$$(f \circ g) \circ h = f \circ (g \circ h)$$

(two functions are equal if for every input, they give the same output)

\textbf{Prove it!}
Left Inverse of a Function

- \( g : B \rightarrow A \) is a **left inverse** of \( f : A \rightarrow B \) if \( g ( f (a) ) = a \) for all \( a \in A \)
  - If you follow the function from the domain to the codomain, the left inverse tells you how to go back to where you started
Left Inverse of a Function

- $g : B \rightarrow A$ is a left inverse of $f : A \rightarrow B$ if $g(f(a)) = a$ for all $a \in A$
  
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Left Inverse of a Function

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Right Inverse of a Function

• $h : B \rightarrow A$ is a **right inverse** of $f : A \rightarrow B$ if $f( h(b) ) = b$ for all $b \in B$

  - If you're trying to get to a destination in the codomain, the right inverse tells you a possible place to start
Right Inverse of a Function

• $h : B \rightarrow A$ is a right inverse of $f : A \rightarrow B$ if $f( h(b) ) = b$ for all $b \in B$

  - If you're trying to get to a destination in the codomain, the right inverse tells you a possible place to start
Right Inverse of a Function

- \( h : B \rightarrow A \) is a **right inverse** of \( f : A \rightarrow B \) if \( f( h(b) ) = b \) for all \( b \in B \)
  - If you're trying to get to a destination in the codomain, the right inverse tells you a possible place to start
Right Inverse of a Function

- $h : B \rightarrow A$ is a right inverse of $f : A \rightarrow B$ if $f(h(b)) = b$ for all $b \in B$
  - If you're trying to get to a destination in the codomain, the right inverse tells you a possible place to start.
Note the subtle difference!

- The **left inverse** tells you how to *exactly* retrace your steps, *if* you managed to get to a destination
  - “Some places might be unreachable, but I can always put you on the return flight”

- The **right inverse** tells you where you *might* have come from, for *any* possible destination
  - “All places are reachable, but I can't put you on the return flight because I don't know exactly where you came from”
Factoid of the Day #2

Left and right inverses need not exist, and need not be unique

can you come up with some examples?
Left inverse $\Leftrightarrow$ Injective

- **Theorem:** A function is **injective** (one-to-one) iff it has a **left inverse**

- **Proof** ($\Leftarrow$): Assume $f: A \rightarrow B$ has left inverse $g$
  
  - If $f(x) = f(y)$ ...
  
  - ... then $g(f(x)) = g(f(y))$ (any fn maps equals to equals)
  
  - ... i.e. $x = y$ (since $g$ is a left inverse)
  
  - Hence $f$ is injective
Left inverse ⇔ Injective

- **Theorem:** A function is *injective* (one-to-one) iff it has a *left inverse*

- **Proof** (⇒): Assume $f : A \to B$ is injective
  - Pick any $a_0$ in $A$, and define $g$ as
    $$g(b) = \begin{cases} 
    a & \text{if } f(a) = b \\
    a_0 & \text{otherwise}
    \end{cases}$$
  - This is a well-defined function: since $f$ is injective, there can be at most a single $a$ such that $f(a) = b$
  - Also, if $f(a) = b$ then $g(f(a)) = a$, by construction
  - Hence $g$ is a left inverse of $f$
Right inverse $\iff$ Surjective

**Theorem:** A function is **surjective** (onto) iff it has a right inverse

**Proof** ($\iff$): Assume $f: A \to B$ has right inverse $h$

- For any $b \in B$, we can apply $h$ to it to get $h(b)$
- Since $h$ is a right inverse, $f(h(b)) = b$
- Therefore every element of $B$ has a preimage in $A$
- Hence $f$ is surjective
Right inverse ⇔ Surjective

• **Theorem:** A function is surjective (onto) iff it has a right inverse

• **Proof** ($\Rightarrow$): Assume $f : A \rightarrow B$ is surjective
  
  – For every $b \in B$, there is a non-empty set $A_b \subseteq A$ such that for every $a \in A_b, f(a) = b$ (since $f$ is surjective)
  
  – Define $h : b \mapsto$ an arbitrary element of $A_b$
  
  – Again, this is a well-defined function since $A_b$ is non-empty (and assuming the “axiom of choice”!)
  
  – Also, $f(h(b)) = b$ for all $b \in B$, by construction
  
  – Hence $h$ is a right inverse of $f$
Recap: Left and Right Inverses

- A function is *injective* (one-to-one) iff it has a *left inverse*
- A function is *surjective* (onto) iff it has a *right inverse*
Factoid for the Day #3

If a function has both a left inverse and a right inverse, then the two inverses are identical, and this common inverse is unique

(Prove!)

This is called the two-sided inverse, or usually just the inverse $f^{-1}$ of the function $f$
Bijection and two-sided inverse

- A function $f$ is bijective iff it has a two-sided inverse

  **Proof** ($\Rightarrow$): If it is bijective, it has a left inverse (since injective) and a right inverse (since surjective), which must be one and the same by the previous factoid

  **Proof** ($\Leftarrow$): If it has a two-sided inverse, it is both injective (since there is a left inverse) and surjective (since there is a right inverse). Hence it is bijective.