Independence and Conditional Probability

CS 2800: Discrete Structures, Spring 2015

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Independence of Events

Two events A and B in a probability space are independent if and only if

$$\mathrm{P}(A \cap B) = \mathrm{P}(A)\,\mathrm{P}(B)$$

Mathematical **definition** of independence

WTF?

Why does this even make sense?

Independence of Events

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if and only if

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(assuming $P(A) \neq 0$)

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if and only if

conditional probability

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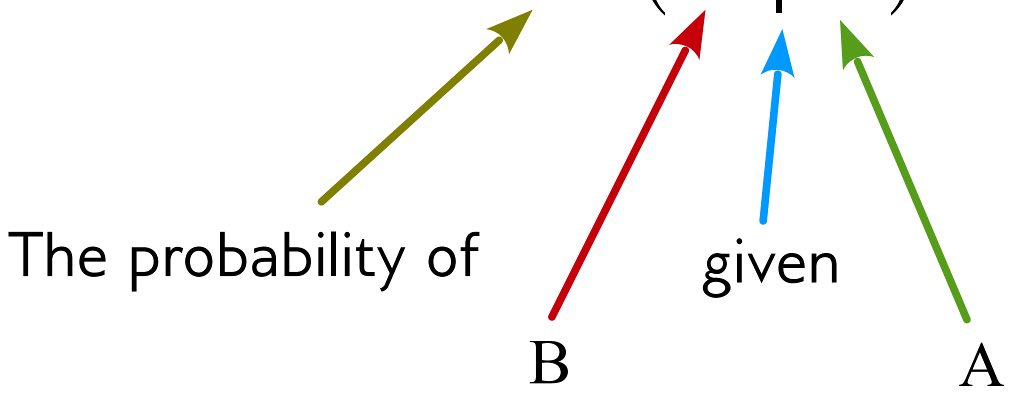

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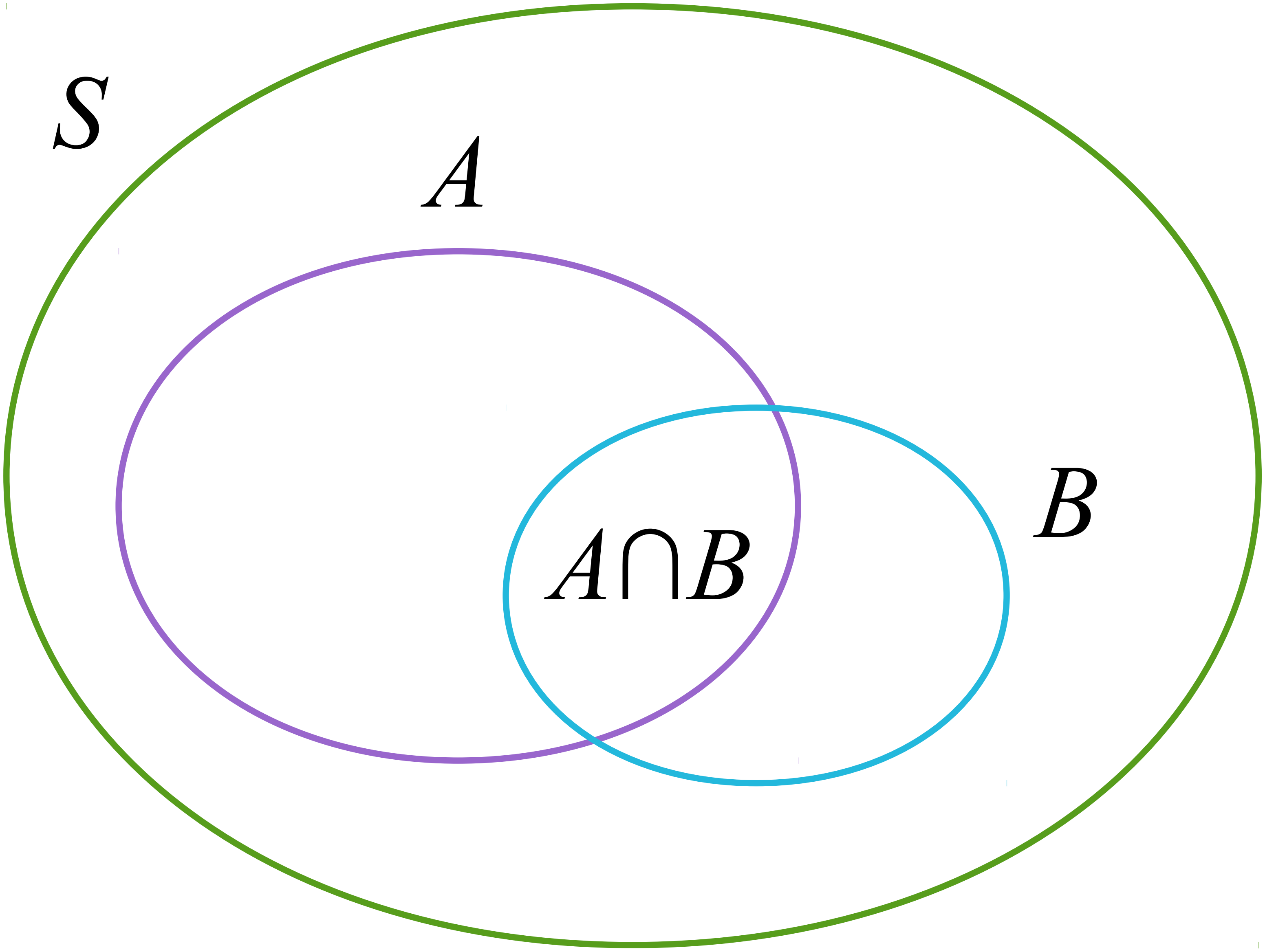
WTF #2?

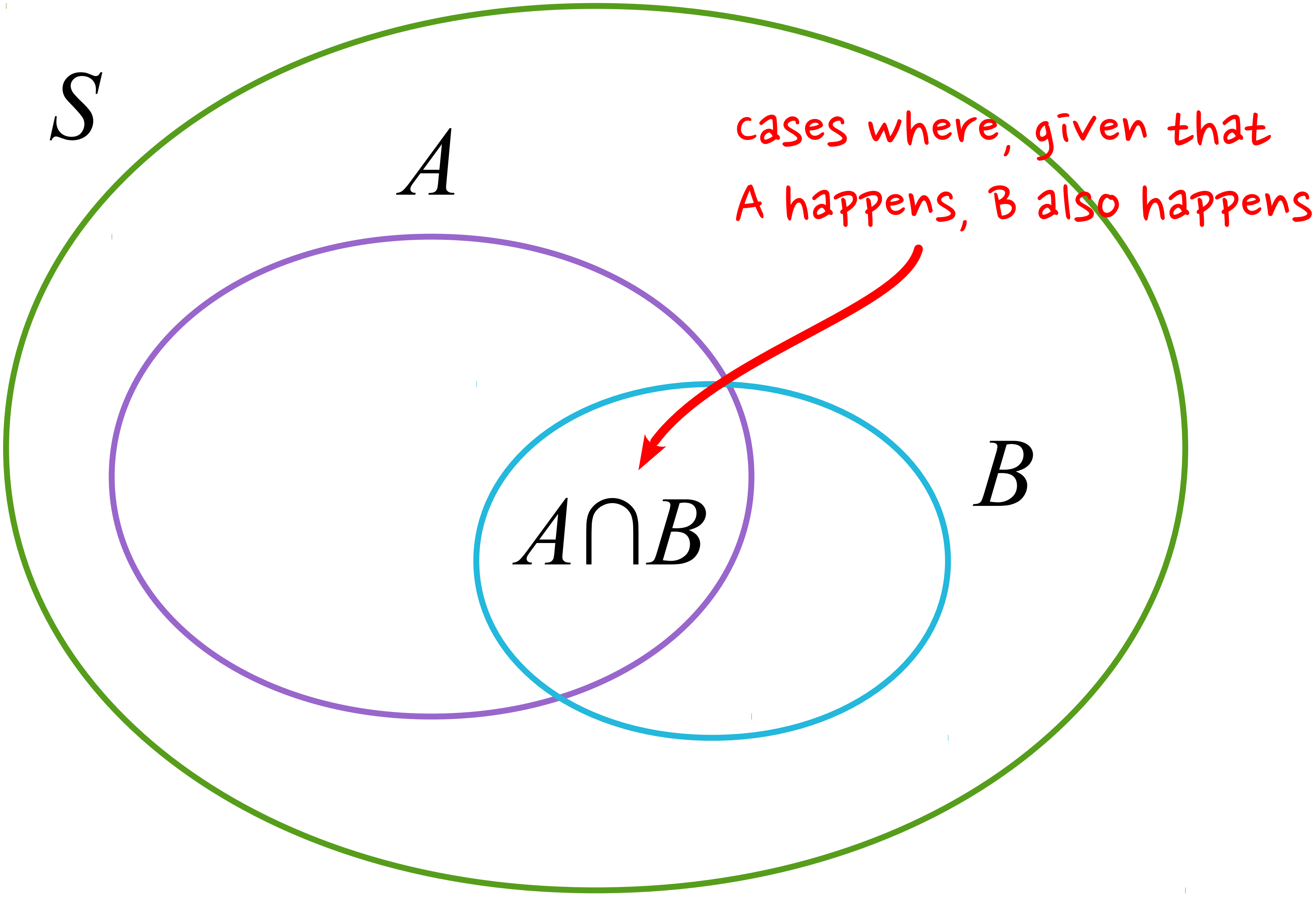
Why does this make sense?

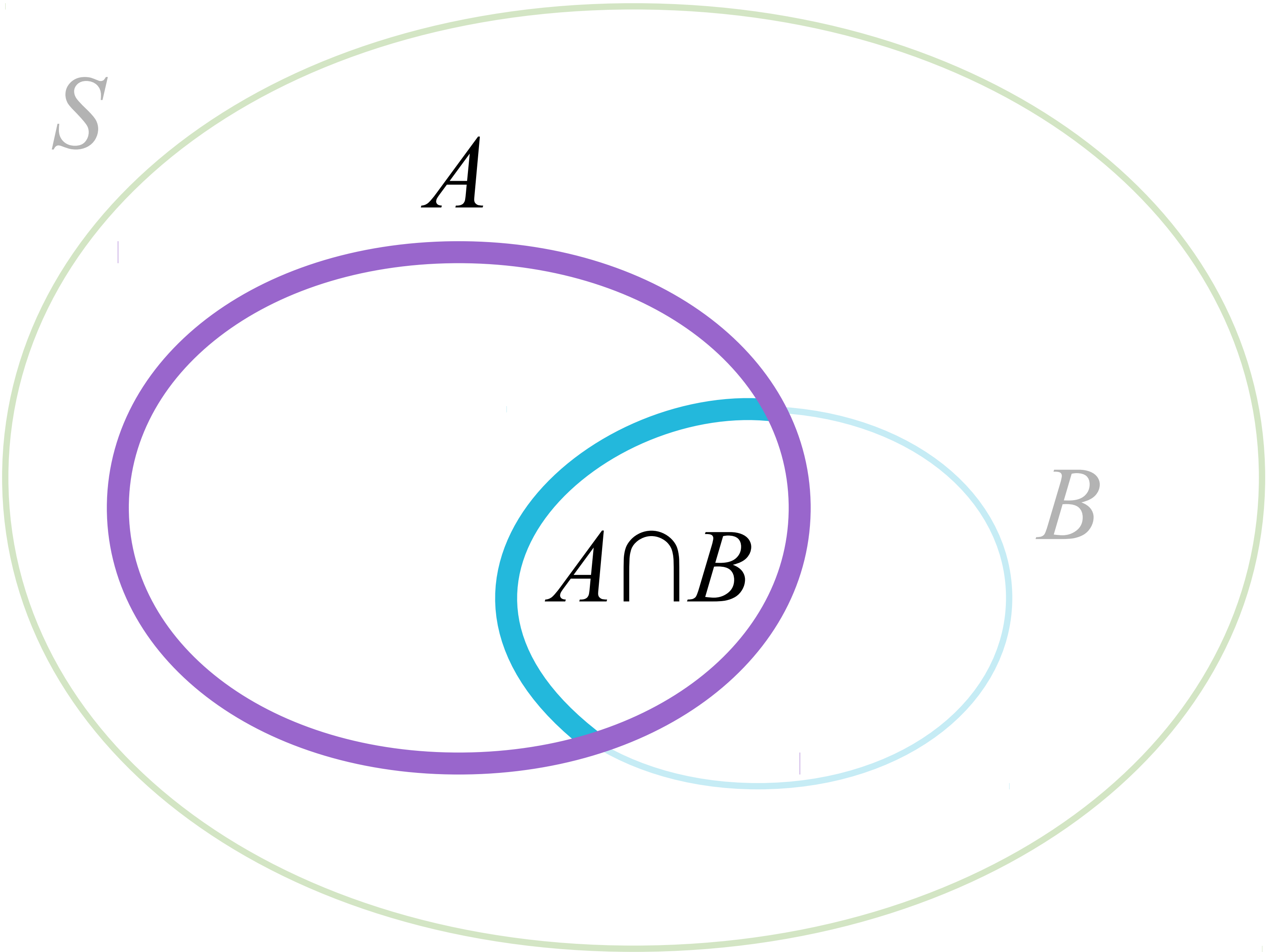
Intuitively, $\mathrm{P}(B \mid A)$ is the probability that event B occurs, given that event A has already occurred

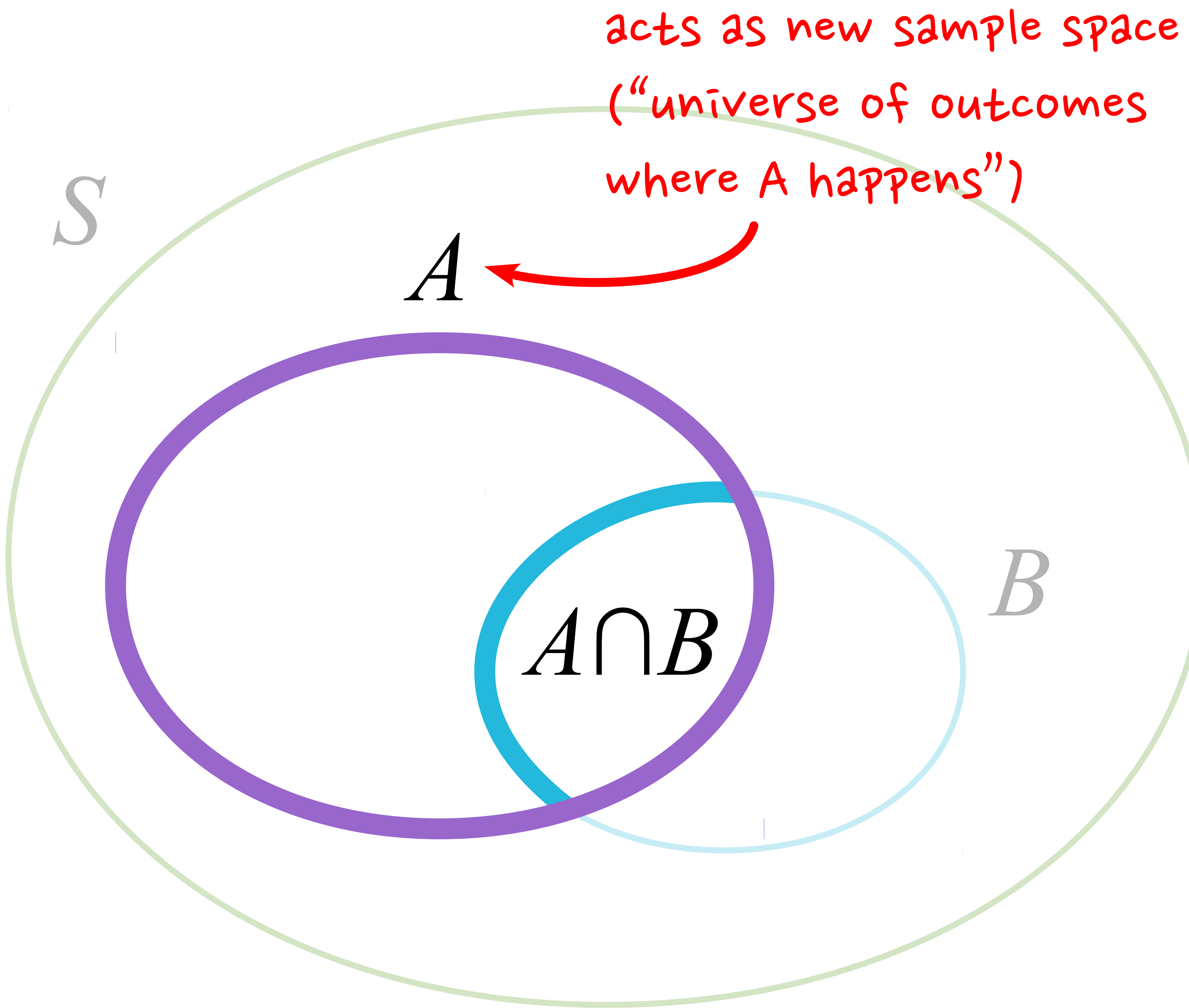
(This is NOT the formal math definition)

(A and B need not actually occur in temporal order)


S
A
B
A∩B


S
A
B
A∩B
cases where, given that
A happens, B also happens


S
A
$A \cap B$
B


acts as new sample space
("universe of outcomes
where A happens")
S
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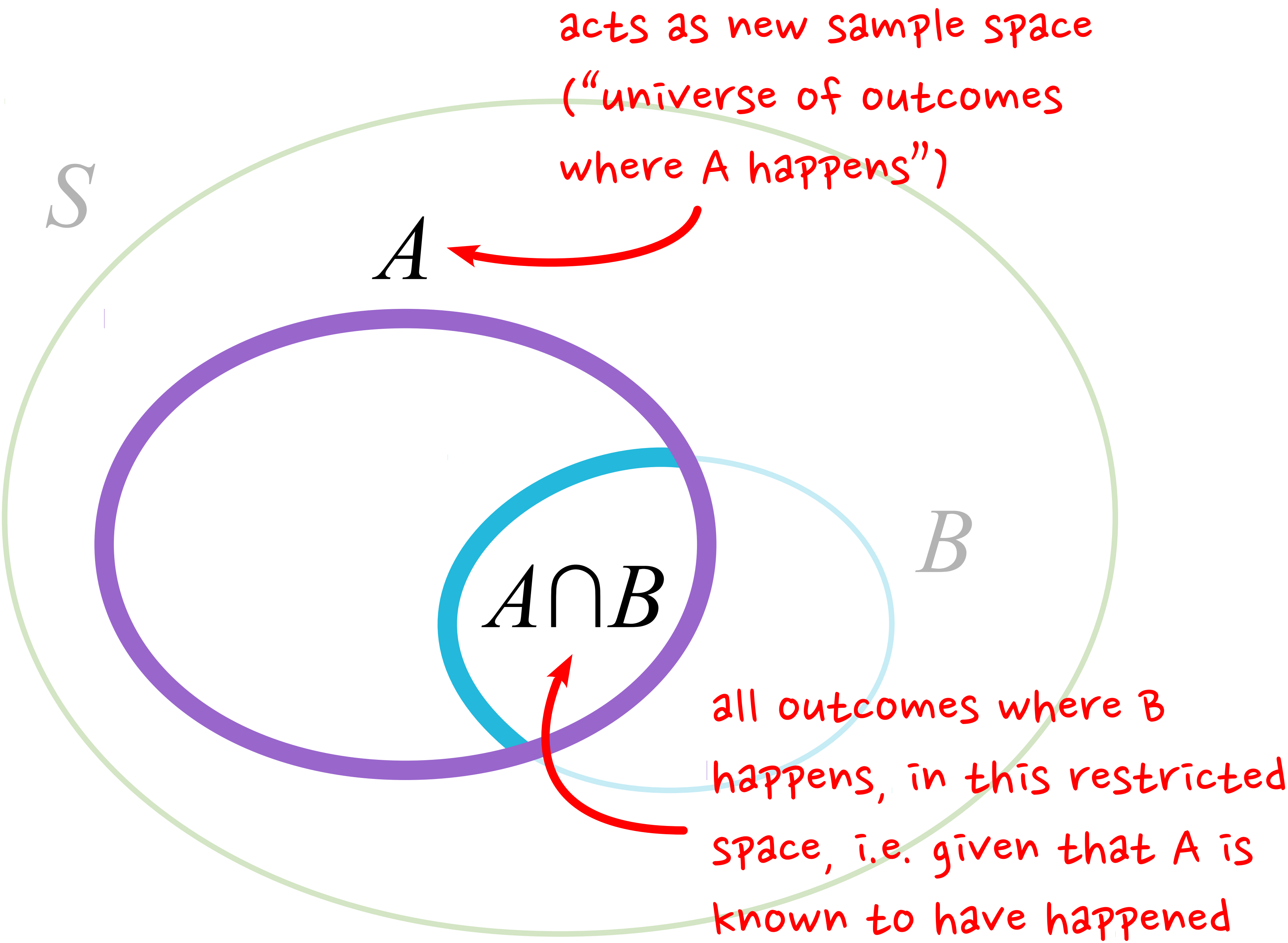
acts as new sample space
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S
A
$A \cap B$
B
all outcomes where B
happens, in this restricted
space, i.e. given that A is
known to have happened

Thought for the Day #1

If the conditional probability $\mathrm{P}(B \mid A)$ is defined as $\mathrm{P}(A \cap B) \,/\, \mathrm{P}(A)$, and $\mathrm{P}(A) \neq 0$, then show that (A, Q), where $\mathrm{Q}(B) = \mathrm{P}(B \mid A)$, is a valid probability space satisfying Kolmogorov's axioms.

Independence of Events

$$P(A \cap B) \;=\; P(B \mid A)\; P(A)$$

(by definition)

$$P(A \cap B) \;=\; P(B)\; P(A)$$

(if independent)

Independence of Events

In other words, assuming $\mathrm{P}(A) \neq 0$, A and B are independent if and only if

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(Note: A and B can be swapped, if $\mathrm{P}(B) \neq 0$)

Bayes' Theorem

Assuming $\mathrm{P}(A), \mathrm{P}(B) \neq 0$,

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

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(by definition of conditional probability)

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Bayes' Theorem

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Assuming $\mathrm{P}(A), \mathrm{P}(B) \neq 0$,

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Posterior probability of A, given evidence B

since $\mathrm{P}(A \mid B)\,\mathrm{P}(B) = \mathrm{P}(A \cap B) = \mathrm{P}(B \mid A)\,\mathrm{P}(A)$
(by definition of conditional probability)

How do we estimate $\mathrm{P}(B)$?

- Theorem of Total Probability (special case):

If $\mathrm{P}(A) \neq 0$ or 1,

$$\begin{aligned}
\mathrm{P}(B) &= \mathrm{P}((B \cap A) \,\cup\, (B \cap A')) \\
&= \mathrm{P}(B \cap A) + \mathrm{P}(B \cap A') && \text{(Axiom 3)} \\
&= \mathrm{P}(B \mid A)\,\mathrm{P}(A) + \mathrm{P}(B \mid A')\,\mathrm{P}(A') && \text{(Definition of conditional probability)}
\end{aligned}$$

Example: Medical Diagnosis

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- Suppose:
  - **1 in 1000** people carry a disease, for which there is a pretty reliable test
  - Probability of a false negative (carrier tests negative) is **1%** (so probability of carrier testing positive is **99%**)
  - Probability of a false positive (non-carrier tests positive) is **5%**
- A person just tested positive. What are the chances (s)he is a carrier of the disease?

Example: Medical Diagnosis

- Priors:
  - $P(\textit{Carrier}) = 0.001$
  - $P(\textit{NotCarrier}) = 1 - 0.001 = 0.999$

Example: Medical Diagnosis

- Priors:
  - $P(Carrier)$ = 0.001
  - $P(NotCarrier)$ = 1 – 0.001 = 0.999
- Conditional probabilities:
  - $P(Positive \mid Carrier)$ = 0.99
  - $P(Positive \mid NotCarrier)$ = 0.05

Example: Medical Diagnosis

$$P(\textit{Carrier} \mid \textit{Positive})$$

$$= \frac{P(\textit{Positive} \mid \textit{Carrier})\ P(\textit{Carrier})}{P(\textit{Positive})}$$

(by Bayes' Theorem)

Example: Medical Diagnosis

$$P(Carrier \mid Positive)$$

$$= \frac{P(Positive \mid Carrier)\ P(Carrier)}{P(Positive)}$$

(by Bayes' Theorem)

$$= \frac{P(Positive \mid Carrier)\ P(Carrier)}{\left(\begin{array}{c} P(Positive \mid Carrier)\ P(Carrier) \\ +\ P(Positive \mid NotCarrier)\ P(NotCarrier) \end{array} \right)}$$

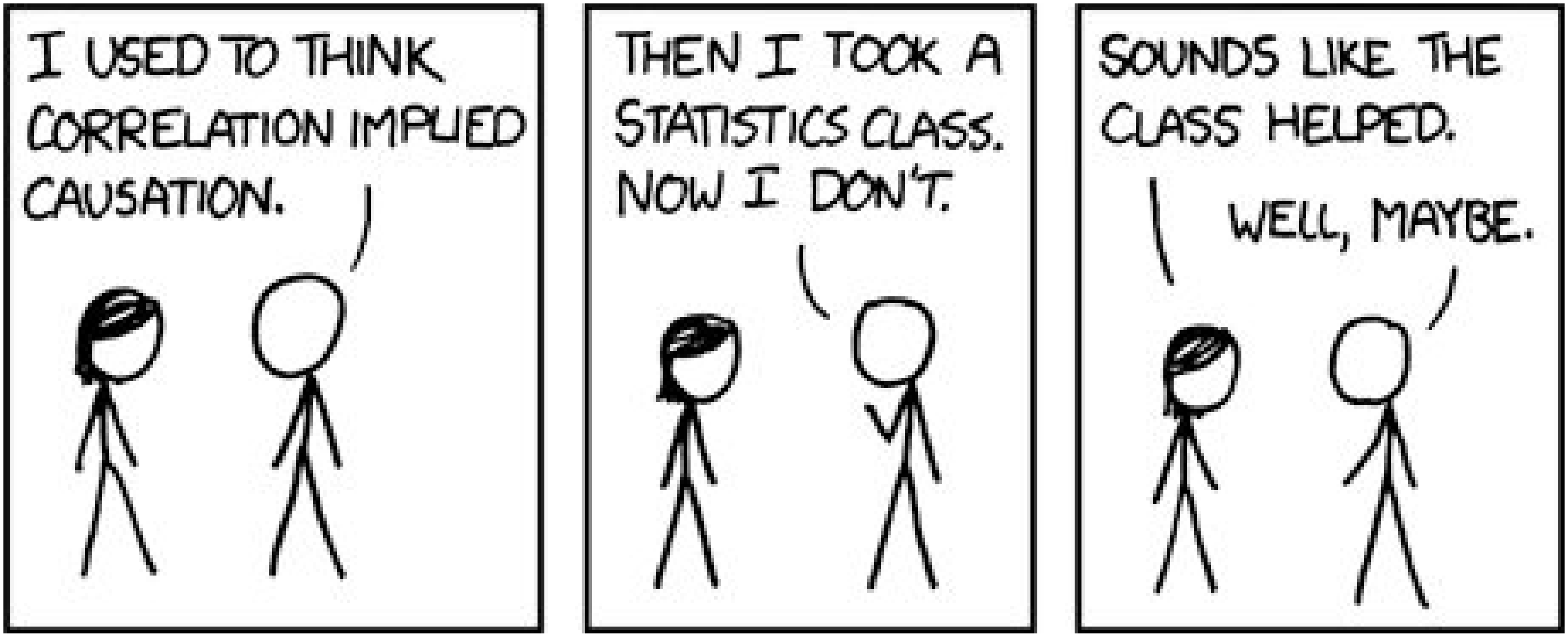
(by Theorem of Total Probability)

Example: Medical Diagnosis

$$\frac{P(\textit{Positive} \mid \textit{Carrier})\ P(\textit{Carrier})}{\left(\begin{array}{c} P(\textit{Positive} \mid \textit{Carrier})\ P(\textit{Carrier}) \\ +\ P(\textit{Positive} \mid \textit{NotCarrier})\ P(\textit{NotCarrier}) \end{array}\right)}$$

$$= \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.05 \times 0.999}$$

$$= 0.0194$$


I USED TO THINK CORRELATION IMPLIED CAUSATION.
THEN I TOOK A STATISTICS CLASS. NOW I DON'T.
SOUNDS LIKE THE CLASS HELPED.
WELL, MAYBE.

XKCD