

# Principle of Mathematical Induction

If it is known that

- (1) some statement is true for  $n = 1$
- (2) assumption that statement is true for  $n$  *implies* that the statement is true for  $(n + 1)$

then the statement is true for *all* positive integers

## Modifications of the Principle of Mathematical Induction

- If it is known that

- (1) some statement is true for  $n = n_0$  (positive integer)
- (2) assumption that statement is true for  $n$  *implies* that the statement is true for  $(n + 1)$

then the statement is true for *all* positive integers greater or equal to  $n_0$

- If it is known that

- (1) some statement is true for  $n = 1$
- (2) assumption that statement is true for all positive integers  $k$ ,  $1 \leq k \leq n$  implies that the statement is true for  $(n + 1)$

then the statement is true for all positive integers

- (*Backward induction*)

If it is known that

- (1) some statement is true for  $n = 1$
- (2) assumption that statement is true for  $n > 1$  implies that the statement is true for  $2n$  and  $(n - 1)$

then the statement is true for all positive integers

## Mathematical Induction in Algebra

1. Prove that any positive integer  $n > 1$  is either a prime or can be represented as product of primes factors.
2. Set  $S$  contains all positive integers from 1 to  $2n$ . Prove that among any  $n + 1$  numbers chosen from  $S$  there are two numbers such that one is a factor of the other.
3. Prove that if  $(x + 1/x)$  is integer then  $(x^n + 1/x^n)$  is also integer for any positive integer  $n$ .
4. For sequence of Fibonacci numbers  $u_1 = 1, u_2 = 1, u_{k+1} = u_k + u_{k-1}, k = 2, 3, \dots$  prove the formula  $u_{k+m} = u_{k-1}u_m + u_k u_{m+1}$
5. Prove the following identities:
  - (a)  $1^2 + 2^2 + 3^2 + \dots + n^2 = n(n+1)(2n+1)/6$
  - (b)  $1^3 + 2^3 + 3^3 + \dots + n^3 = n^2(n+1)^2/4$
  - (c)  $1 \times 2 \times 3 + 2 \times 3 \times 4 + \dots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$
  - (d)  $1 \times 1! + 2 \times 2! + \dots + n \times n! = (n+1)! - 1$
6. Prove the following divisibilities:
  - (a)  $6/n^3 + 5n$
  - (b)  $7/(6^{2n-1} + 1)$
  - (c)  $3^{n+1}/2^{3^n} + 1$
7. Prove the following inequalities:
  - (a)  $1/(n+1) + 1/(n+2) + 1/(n+3) + \dots + 1/2n > 13/24$  ( $n > 1$ )
  - (b)  $1/1^2 + 1/2^2 + 1/3^2 + \dots + 1/n^2 < 2$
  - (c)  $(x_1 + \dots + x_n)/n \geq (x_1 \times \dots \times x_n)^{1/n}$ , where  $x_1, \dots, x_n$  are positive numbers

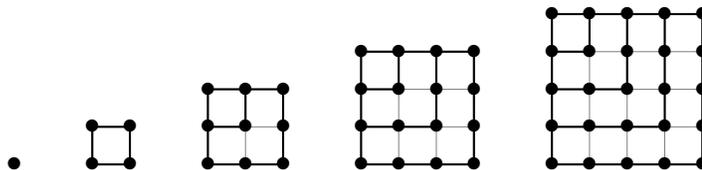
## Mathematical Induction in Geometry

1. Prove that a square may be cut in any number of smaller squares greater than 5 (no leftovers).
2.  $N$  straight lines in general position divide a plane into several regions. Find the number of these regions. (Straight lines are said to be in general position if any pair of lines intersect, and no three lines pass through the same point).
3.  $N$  circles divide a plane into several regions. Find the number of regions, if every two circles intersect in two points and no three circles pass through the same point.
4. From a square  $2^{13} \times 2^{13}$  one cell is cut out. Prove that one may pave the resulting figure by 3 cells angles.
5. Several straight lines and circles are drawn on a plane. Prove that one can paint the regions, created in result of intersection of the lines into black and white colors so that adjacent regions are of the different colors.
6. Several straight lines split a plane into several regions. Each line has a shadow on one of its sides. Prove that there is a region completely covered by each of its shadows.

## Mathematical Induction in Combinatorics

1. Suppose  $S$  is a set with  $n$  elements. Prove, that the set of all subsets  $S$  has  $2^n$  elements.
2. On the first cell of a strip of squares sits a grasshopper. Each minute it jumps to the right either to the next cell or on the second to next cell. Find the number of ways it can reach the  $n$ -th cell.
3. A traveller came to a country inn. He had no money, but did have a chain made of gold rings. The host of the inn agreed to take the gold rings as a payment, one ring per day (payments must be made daily). To accomplish the task, the traveller is allowed to break  $n$  rings. Find the maximal duration of the stay (in terms of  $n$ ).

4. Pentagonal numbers. To define sequence of pentagonal numbers  $P_m(n)$  one needs to pack balls into regular  $m$ -gons with  $n$  balls on each side. For example,  $P_4(1)=1$ ,  $P_4(2)=4$ ,  $P_4(3)=9$ ,  $P_4(4)=16$ ,  $P_4(5)=25$ . (Look at pictures below)



Establish the formula for  $P_m(n)$ , where  $m \geq 3$ .

5. Establish and prove an estimate for minimal number of operations to solve the “The Towers of Hanoi” puzzle (problem 1, Mathematical Induction in Processes)

## Mathematical Induction in Processes

1. “The Towers of Hanoi” is a puzzle with 3 nails and 7 rings, all of different sizes. Initially all rings are on the same nail in decreasing order from the bottom to the top. The procedure is removing the top ring from any nail and placing it on another nail. It is not allowed to place a bigger ring on the top of a smaller one. Is it possible to replace all the rings from one nail onto another by applying this procedure? What is the minimal number of operations needed to solve the puzzle? Establish and prove the formula.
2. Each of  $n$  identical in shape jars is filled with a paint to  $(n-1)/n$  of its volume. No two jars contain the same kind of the paint. It is allowed to pour any amount of paint from one jar to another. Could one get the same mixture in all jars? Paint is not disposable and there is no other empty vessels.
3. A band of  $n$  pirates divide a pile of treasures between themselves. Each wants to be sure that he gets no less than  $1/n$ -th of the pile. Find a fair way to split the pile between them (so that anyone can blame no one except maybe himself). The pirates’ opinions on size of the piles could be different.
4. (Tournament of Towns 2005, Fall Round) There are 1000 pots each containing varying amounts of jam, not more than  $1/100$ -th of the total. Each day, exactly 100 pots are to be chosen, and from each pot, the same amount of jam is eaten. Prove that it is possible to eat all the jam in a finite number of days.