• There are a lot of niceties in defining what a set should be, but for now we’ll leave that to the more esoteric regions of the foundations of mathematics, and be satisfied with the inherently problematic definition that a set is a collection of objects defined by some rule (i.e., the rule tells us whether any given objects should or should not be in the collection).*

• Unless we say otherwise, we’ll treat sets like \{ a, b, c, a, b, a, d \} and \{ a, b, d, c \} as being equal, i.e., the order of listing members is irrelevant, and any repetition of the same element in the description doesn’t add in multiple copies of it. **

• As in our discussion of logic, so again for sets we’ll start by looking at ways of manipulating sets. *** There are two particular sets of importance: the empty set, denoted \emptyset, and the universe (comprising everything under potential consideration, which we’ll choose to denote by \mathcal{U} ).

• There are some natural ways of acting on sets:
  • If \( A \) and \( B \) are two sets, then \( A - B \) is the set of all things which are in \( A \) but not in \( B \) and could write this formally as \( A - B = \{ x \in A \mid x \notin B \} \), the set difference. Note that \( x \in A \) denotes that \( x \) is a member of the set \( A \), and the vertical line in this context means ‘such that’.
  • \( A \cup B \) is the union of \( A \) and \( B \), formally \( A \cup B = \{ x \in \mathcal{U} \mid x \in A \lor x \in B \} \).
  • \( A \cap B \) is the intersection of \( A \) and \( B \), formally \( A \cap B = \{ x \in \mathcal{U} \mid x \in A \land x \in B \} \).
  • \( A^c \) is the complement of \( A \), formally \( A^c = \mathcal{U} - A \).
  • \( A + B \) is the symmetric difference of \( A \) and \( B \), formally \( A + B = (A - B) \cup (B - A) \).

* For an elaboration of the problems, look up Russell’s paradox, and think in terms of such collections being able to be a bit too large ... the formal ways of fixing this are essentially ways of ensuring that there’s a restriction on how big a set can be -- infinite is perfectly fine however.

** In the case that this latter aspect is permitted, we’ll be explicit in calling such animals multisets.

*** It’s worth noting that whenever any somewhat general results are stated, it’s often helpful to have a few specific examples to play with so that you can start to get a feel for what the results mean in practice. Proof by example is a common attempt by many students, but only works if you test it for every conceivable example(!!!), but it is nevertheless a great way to help decide if you actually believe the result.
• We write $A \subseteq B$ to mean that $A$ is a subset of $B$, meaning that everything in $A$ is also in $B$. Note that for two sets $A$ and $B$ to be equal means that $A \subseteq B$ and $B \subseteq A$.

• There are a number of quasi-algebraic manipulations we can do with sets:

1. $A \cup (B \cup C) = (A \cup B) \cup C$
2. $A \cup B = B \cup A$
3. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. $A \cup \emptyset = A$
5. $A \cup A^c = U$
6. $A \cup B = A$ for all sets $A \Rightarrow B = \emptyset$
7. $A \cup B = U$ and $A \cap B = \emptyset \Rightarrow B = A^c$
8. $(A^c)^c = A$
9. $\emptyset^c = U$
10. $A \cup A = A$
11. $A \cup U = U$
12. $A \cup (A \cap B) = A$
13. $(A \cup B)^c = A^c \cap B^c$ **

Sample wordy proof for 3:

Let $x \in (A \cup B) \cap (A \cup C)$, then by definition of $\cap$ this means that $x \in A \cup B$ and $x \in A \cup C$. $x \in A \cup B$ means by definition of $\cup$ that $x \in A$ or $x \in B$, or both. $x \in A \cup C$ means by definition of $\cup$ that $x \in A$ or $x \in C$, or both.

So if $x \notin A$ then it has to be that $x \in B$ and $x \in C$. If $x \in A$ then it needn’t be in $B$ or $C$, but no problem occurs if it is.

Hence $x \in A \cup (B \cap C)$, and we’ve shown that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

To show the converse, let’s argue by contradiction.

Suppose that $y \in A \cup (B \cap C)$ but that $y \notin (A \cup B) \cap (A \cup C)$. (*)

$y \in A \cup (B \cap C)$ means by definition of $\cup$ that $y \in A$ or $y \in B \cap C$, or both.

If $y \in A$ then $y \in A \cup B$ and $y \in A \cup C$, which would contradict (*).

So $y \notin A$.

Hence $y \in B \cap C$, which by definition of $\cap$ means that $y \in B$ and $y \in C$.

But then $y \in A \cup B$ and $y \in A \cup C$, again contradicting (*).

Hence (*) is false, and we’ve shown that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Thus we have that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Sample symbolic proof for 3:

$x \in (A \cup B) \cap (A \cup C) \iff (x \in A \cup B) \wedge (x \in A \cup C)$, by definition of $\cap$

$\iff ((x \in A) \lor (x \in B)) \wedge ((x \in A) \lor (x \in C))$, by definition of $\cup$

$\iff ((x \in A) \wedge (x \in A)) \lor ((x \in B) \lor (x \in C))$, by distribution of $\wedge$ and $\lor$

$\iff ((x \in A) \lor (x \in B \cap C))$, by definition of $\cap$

$\iff x \in A \cup (B \cap C)$, by definition of $\cup$

• All of the above have dual versions obtained by swapping $\cup$ and $\cap$, and $\emptyset$ and $U$.

* You might like to try proving them.

** This is one of the De Morgan’s laws ... the other is its dual.