Inclusion-Exclusion

CS 2800: Discrete Structures, Fall 2014

Sid Chaudhuri
ONE DOES NOT SIMPLY
HIT THE SNOOZE BUTTON ONCE
Probability of general celebration in class

- Buses are delayed/cancelled *10%* of the time during the winter
- I oversleep *20%* of the time
Analogous: a serial circuit
Analogous: a serial circuit

 Broken 10\% of the time

 Broken 20\% of the time
What's the probability no current is flowing?

A

Broken 10% of the time

B

Broken 20% of the time
Modeling

- $A$: event that buses are delayed
  - (or first component breaks)
- $B$: event that I oversleep
  - (or second component breaks)
- $Late = A \cup B$: event that I am late
  - (or current is blocked)
Probability of a Union

- Kolmogorov's 3rd Axiom guarantees a simple formula for the probability of the union of mutually exclusive events in a probability space

\[ P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + P(E_3) + \ldots \]
Probability of a Union

- Kolmogorov's 3rd Axiom guarantees a simple formula for the probability of the union of mutually exclusive events in a probability space

\[ P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + P(E_3) + \ldots \]

- But what if the events are not mutually exclusive?
$S$  

$A$  

$A \cap B$  

$B$
Counting Elements

$|A \cup B|$
Counting Elements

\[ |A \cup B| = |A \cup (B \setminus A)| \]
Counting Elements

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\[ = |A| + |B \setminus A| \]
Counting Elements

\[|A \cup B| = |A \cup (B \setminus A)|\]
\[= |A| + |B \setminus A|\]
\[= |A| + |B \setminus A| + |A \cap B| - |A \cap B|\]
Counting Elements

\[ |A \cup B| = |A \cup (B \setminus A)| \]

\[ = |A| + |B \setminus A| \]

\[ = |A| + (|B \setminus A| + |A \cap B|) - |A \cap B| \]
\[ |A \cup B| = |A \cup (B \setminus A)| \]
\[ = |A| + |B \setminus A| \]
\[ = |A| + |B \setminus A| + |A \cap B| - |A \cap B| \]
\[ = |A| + |(B \setminus A) \cup (A \cap B)| - |A \cap B| \]
Counting Elements

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Counting Elements

$$|A \cup B| = |A \cup (B \setminus A)|$$

$$= |A| + |B \setminus A|$$

$$= |A| + |B \setminus A| + |A \cap B| - |A \cap B|$$

$$= |A| + |(B \setminus A) \cup (A \cap B)| - |A \cap B|$$

$$= |A| + |B| - |A \cap B|$$

A similar result holds for probabilities
The Inclusion-Exclusion Principle
(for two events)

For two events $A, B$ in a probability space:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
The Inclusion-Exclusion Principle
(for two events)

For two events $A, B$ in a probability space:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Don't use this to “prove” Kolmogorov's Axioms!!!
The Inclusion-Exclusion Principle
(for two events)

Proof:

\[ P(A \cup B) = P(A \cup (B \setminus A)) \]  (set theory)
The Inclusion-Exclusion Principle
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\[ P(A \cup B) = P(A \cup (B \setminus A)) \]

\[ = P(A) + P(B \setminus A) \]  

(set theory)  

(mut. excl., so Axiom 3)
The Inclusion-Exclusion Principle
(for two events)

Proof:

\[ P(A \cup B) = P(A \cup (B \setminus A)) \quad \text{(set theory)} \]

\[ = P(A) + P(B \setminus A) \quad \text{(mut. excl., so Axiom 3)} \]

\[ = P(A) + P(B \setminus A) + P(A \cap B) - P(A \cap B) \]

\[ \quad \text{(Adding 0 = P(A \cap B) – P(A \cap B) )} \]
The Inclusion-Exclusion Principle  
(for two events) 

Proof: 

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(set theory) 

\[ = P(A) + P(B \setminus A) \quad \text{(mut. excl., so Axiom 3)} \] 

\[ = P(A) + P(B \setminus A) + P(A \cap B) - P(A \cap B) \]  
(Adding 0 = P(A \cap B) – P(A \cap B) ) 

\[ = P(A) + P((B \setminus A) \cup (A \cap B)) - P(A \cap B) \]  
(mut. excl., so Axiom 3)
The Inclusion-Exclusion Principle
(for two events)

Proof:

\[ P(A \cup B) = P(A \cup (B \setminus A)) \]  
\[ = P(A) + P(B \setminus A) \]  
\[ = P(A) + P(B \setminus A) + P(A \cap B) - P(A \cap B) \]  
\[ (\text{mut. excl., so Axiom 3}) \]

\[ = P(A) + P((B \setminus A) \cup (A \cap B)) - P(A \cap B) \]  
\[ (\text{mut. excl., so Axiom 3}) \]

\[ = P(A) + P(B) - P(A \cap B) \]  
\[ (\text{set theory}) \]
Will I be late for class?

- $P(\text{Late}) = P(A \cup B)$
Will I be late for class?

- $P(\text{Late}) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Will I be late for class?

- \( P(\text{Late}) = P(A \cup B) \)
  
  \[ = P(A) + P(B) - P(A \cap B) \]
  
  \[ = 0.1 + 0.2 - ??? \]
Will I be late for class?

- \( P(\text{Late}) = P(A \cup B) \)
  
  \[
  = P(A) + P(B) - P(A \cap B)
  \]
  
  \[
  = 0.1 + 0.2 - ???
  \]

Let's make the **modeling assumption** 
\( A \) and \( B \) are independent: 
\( P(A \cap B) = P(A) P(B) \)
Will I be late for class?

- $P(Late) = P(A \cup B)$
  
  $= P(A) + P(B) - P(A \cap B)$
  
  $= P(A) + P(B) - P(A) \cdot P(B)$
Will I be late for class?

- $P(Late) = P(A \cup B)$
  
  $= P(A) + P(B) - P(A \cap B)$
  
  $= P(A) + P(B) - P(A) \cdot P(B)$
  
  $= 0.1 + 0.2 - 0.1 \times 0.2$
Will I be late for class?

- $P(Late) = P(A \cup B)$
  
  $= P(A) + P(B) - P(A \cap B)$
  
  $= P(A) + P(B) - P(A) \cdot P(B)$
  
  $= 0.1 + 0.2 - 0.1 \times 0.2$

  $= 0.28$
The Inclusion-Exclusion Principle
(for three events)

For three events $A, B, C$ in a probability space:

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) \]
\[ - P(A \cap B) - P(B \cap C) - P(C \cap A) \]
\[ + P(A \cap B \cap C) \]
The Inclusion-Exclusion Principle

For events $A_1, A_2, A_3, \ldots A_n$ in a probability space:

$$
P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(\bigcap_{i=1}^{n} A_i)
$$
The Inclusion-Exclusion Principle

For events $A_1, A_2, A_3, \ldots, A_n$ in a probability space:

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) + \sum_{1 \leq i < j < k \leq n} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(\bigcap_{i=1}^{n} A_i)$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{\substack{I \subseteq \{1, 2, \ldots, n\} \mid |I|=k}} P(\bigcap_{i \in I} A_i)$$