CS 2800: Discrete Structures (Fall '11) Oct.26, 2011

Induction

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Concept of Inductive Proof

When you think of induction, one of the best analogies to think about is **ladder**. When you climb up the ladder, you have to step on the lower step and need to go up based on it. After we climb up the several steps, we can go up further by **assuming** that the step you are stepping on exists. With the terms we have covered in class we can make such analogies.

1. Base Case : The first step in the ladder you are stepping on
2. Induction Hypothesis : The steps you are assuming to exist

   Weak Induction : The step that you are currently stepping on

   Strong Induction : The steps that you have stepped on before including the current one
3. Inductive Step : Going up further **based on** the steps we assumed to exist

Components of Inductive Proof

Inductive proof is composed of 3 major parts : Base Case, Induction Hypothesis, Inductive Step. When you write down the solutions using induction, it is **always** a great idea to think about this template.

1. Base Case : One or more particular cases that represent the most basic case. (e.g. n=1 to prove a statement in the range of positive integer)
2. Induction Hypothesis : Assumption that we would like to be based on. (e.g. Let's assume that P(k) holds)
3. Inductive Step : Prove the next step based on the induction hypothesis. (i.e. Show that Induction hypothesis P(k) implies P(k+1))

Weak Induction, Strong Induction

This part was not covered in the lecture explicitly. However, it is always a good idea to keep this in mind regarding the differences between weak induction and strong induction.

The difference between weak induction and strong indcution only appears in induction hypothesis. In weak induction, we only assume that particular statement holds at *k-th* step, while in strong induction, we assume that the particular statment holds at all the steps from the base case to *k-th* step.

Weak Induction Example

Prove the following statement is true for all integers n.The staement P(n) can be expressed as below :

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{1}$$

1. **Base Case** : Prove that the statement holds when n = 1

$$\sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1 \tag{2}$$

2. **Induction Hypothesis** : Assume that the statment holds when n = k

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \tag{3}$$

3. **Inductive Step** : Prove that the statement holds when when n = k+1 using the assumption above.

   In the exam, many of you have struggled in this part. Please pay close attention to how this suggested inductive step uses induction hypothesis for reasoning.

   We already assumed the following fact :

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \tag{4}$$

   Now we would like to consider the case of finding :

$$\sum_{i=1}^{k+1} i \tag{5}$$

   We can interpret the formula above in this way :

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \tag{6}$$

   From the induction hypothesis above, the right hand side of the equation above can be written as :

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) \tag{7}$$

   We can simplify this re-formatted right-hand side :

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2} \tag{8}$$

   Now, we can conclude that when i = k+1, the proposition we are trying to prove holds, because under this case, n = k+1 and n+1 = k+2. Therefore, the proposition holds for all integers n.

Strong Induction Example

Prove by induction that every integer greater than or equal to 2 can be factored into primes. The statement P(n) is that an integer n greater than or equal to 2 can be factored into primes.

1. **Base Case** : Prove that the statement holds when n = 2

   We are proving P(2). 2 itself is a prime number, so the prime factorization of 2 is 2. Trivially, the statement P(2) holds.

2. **Induction Hypothesis** : Assume that **for all** integers less than or equal to k, the statement holds.

   Note : In the previous example, the assumption was only about the case when n = k.

3. **Inductive Step** : Consider the number k+1.

   Case 1 : k+1 is a prime number.

   When k+1 is a prime number, the number is a prime factorization of itself. Therefore, the statement P(k+1) holds.

   Case 2 : k+1 is *not* a prime number.

   We know that k+1 is a composite, so $k+1 = p \times q (p, q \in \mathbb{Z}^+)$. Intuitively, we can conclude that p and q are less than or equal to k+1.

   From the induction hypothesis stated above, **for all** integers less than or equal to k, the statement holds, which means both p and q can be expressed as prime factorizations. In this sense, because k+1 is a product of p and q, by multiplying the prime factorizations of p and q, we can get the prime factorization for k+1 as well.

   Therefore, the statement that every integer greater than or equal to 2 can be factored into primes holds for all such integers.

The common mistake in this question was to prove the **Case 2** *in the inductive step without using induction hypothesis by dividing the cases further into even number and odd number, etc. It works, but does not fit into the notion of inductive proof that we wanted you to learn. For inductive step in inductive proof, you* **must** *reason your argument based on induction hypothesis you yourself state.*

There are several reasonable questions that might appear while reading the proof above.

1. Why is it sufficient to prove only n=2 as a base case for this question?

   In this question, generally most questions using strong induction, for each step, as stated in the induction hypothesis, we assume that the statement holds for **all** previous steps. To prove P(3) works, we assume P(2) works; to prove P(4) works, we assume P(2), P(3) work. In this sense, prove only the case of n=2 is sufficient enough.

2. Still, for prime numbers which do not have 2 as their factors, don't we need to prove their cases separately?

   Not really; those numbers are dealt in the **Case 1** in the inductive step. If you try to prove all these possible prime number cases, you need to do so with brute-force scheme, which means you need to prove the statement on every single prime number greater than 2.

Summary

Template of Inductive Proof

1. Base Case : **Prove** the most basic case.
2. Induction Hypothesis : **Assume that** the statement holds for some k or for all numbers less than or equal to k.
3. Inductive Step : **Prove** the statement holds for the next step **based on induction hypothesis**.

Checklist

1. Check whether you proved all necessary base cases! Base case is not necessarily one case (sometimes more than one).
2. Check you marked three components of inductive proof correctly.
3. Check you used induction hypothesis appropriately for inductive step.