1 Recurrence Relations: Bottom Up

For each of the characteristic equations:

1. Find the corresponding recurrence equation.
2. How many boundary conditions are necessary for a complete solution to $f(n)$?
3. Show that the roots of the characteristic equations raised to the $n^{th}$ power are solutions of the recurrence equation.  Ie $f(n) = r^n$, where $r$ is a root of the corresponding characteristic equation, satisfies the recurrence equation.

The characteristic equations:

$x - 2 = 0$
$x^2 + 2x - 15 = 0$
$8x^3 - 8x^2 + 4x - 1 = 0$

2 Recurrence Relation: Top Down

For each of the recurrence equations:

1. Find the characteristic equation and its roots.
2. The linearly independent set of basis functions.
3. The final closed form solution of $f(n)$
4. Show for $n = 3$ and $n = 4$, the closed form solution and the recurrence equations agree.

The recurrence relations:

$f(n) = 5f(n-1) - 6f(n-2)$ with boundary conditions $f(0) = 0$ and $f(1) = 1.$

$f(n) = 6f(n-1) - 12f(n-2) + 8f(n-3)$ with boundary conditions $f(0) = 1$, $f(1) = 6$, and $f(2) = 32.$
3 Bijective Proof

Here we provide the outline of a proof. You should rewrite all the provided steps and fill in the missing steps marked by ???. For definitions of 1-1(injective), onto(surjective), and isomorphism(bijection) please reference wikipedia.

**Theorem 1.** Given sets $S$ and $T$, and a mapping $f : S \rightarrow T$ that is onto and a mapping $g : T \rightarrow S$ is onto, prove whether or not $S$ and $T$ are isomorphic.

This is not a proof, but to guess whether or not $S$ and $T$ are isomorphic, we can consider their cardinality. If $f$ is an onto mapping from $S$ to $T$:

$$|S| \leq |T|$$

And similarly, if $g$ is an onto mapping from $T$ to $S$:

$$|S| \leq |T|$$

Which implies:

$$|S| = |T|$$

Hence, if $S$ and $T$ are finite sets they have the same cardinality and would be isomorphic. But, we need to construct a rigorous proof to handle finite and infinite sets.

**Proof.** The proof idea: To prove $S$ and $T$ are isomorphic, we will use mappings $f$ and $g$ to construct a 1−1 mapping from $S$ to $T$ and a 1−1 mapping from $T$ to $S$. From class, if we can show both 1−1 mappings exist, this implies there is a bijection between $S$ and $T$. Hence, $S$ and $T$ are isomorphic.

The proof:

We now construct a 1−1 mapping from $T$ to $S$ using $f$:

$$???$$

We now construct a 1−1 mapping from $S$ to $T$ using $g$:

$$???$$

Hence, from class, as there exists a 1-1 mapping from $S$ to $T$ and from $T$ to $S$, there exists a bijection from $S$ to $T$. Hence, $S$ and $T$ are isomorphic. \hfill \Box

4 Mapping Examples

Consider the set up in the previous problem where $f$ is an onto mapping from $S$ to $T$. In the previous problem we show $S$ and $T$ are isomorphic.

1. Give an example of two sets $S$ and $T$ and mapping $f$, where the fact $S$ and $T$ are isomorphic implies that $f$ is also 1−1.

2. Give an example of two sets $S$ and $T$ and mapping $f$, where the fact $S$ and $T$ are isomorphic does not implies that $f$ is also 1−1.