Permutations

A *permutation* of \( n \) things taken \( r \) at a time, written \( P(n, r) \), is an arrangement in a row of \( r \) things, taken from a set of \( n \) distinct things. Order matters.

**Example 6:** How many permutations are there of 5 things taken 3 at a time?

**Answer:** 5 choices for the first thing, 4 for the second, 3 for the third: \( 5 \times 4 \times 3 = 60 \).

- If the 5 things are \( a, b, c, d, e \), some possible permutations are:

  \[
  \begin{align*}
  abc & \quad abd & \quad a be & \quad acb & \quad acd & \quad ace \\
  adb & \quad adc & \quad ade & \quad a eb & \quad a ec & \quad a ed \\
  \cdots
  \end{align*}
  \]

In general

\[
P(n, r) = \frac{n!}{(n - r)!} = n(n - 1) \cdots (n - r + 1)
\]
Combinations

A combination of $n$ things taken $r$ at a time, written $C(n, r)$ or $\binom{n}{r}$ ("$n$ choose $r$") is any subset of $r$ things from $n$ things. Order makes no difference.

**Example 7:** How many ways are there of choosing 3 things from 5?

**Answer:** If order mattered, then it would be $5 \times 4 \times 3$. Since order doesn’t matter,

$$\text{abc, acb, bac, bca, cab, cba}$$

are all the same.

- For way of choosing three elements, there are $3! = 6$ ways of ordering them.

Therefore, the right answer is $(5 \times 4 \times 3)/3! = 10$:

$$\text{abc, abd, abe, acd, ace}$$
$$\text{ade, bcd, bce, bde, cde}$$

In general

$$C(n, r) = \frac{n!}{(n-r)!r!} = \frac{n(n-1) \cdots (n-r+1)}{r!}$$
More Examples

Example 8: How many full houses are there in poker?

- A full house has 5 cards, 3 of one kind and 2 of another.
- E.g.: 3 5’s and 2 K’s.

Answer: You need to find a systematic way of counting:

- Choose the denomination for which you have three of a kind: 13 choices.
- Choose the three: $C(4, 3) = 4$ choices
- Choose the denomination for which you have two of a kind: 12 choices
- Choose the two: $C(4, 2) = 6$ choices.

Altogether, there are:

$$13 \times 4 \times 12 \times 6 = 3744$$ choices
0!

It’s useful to define $0! = 1$.

Why?

1. Then we can inductively define

$$(n + 1)! = (n + 1)n!,$$

and this definition works even taking 0 as the base case instead of 1.

2. A better reason: Things work out right for $P(n, 0)$ and $C(n, 0)$!

How many permutations of $n$ things from $n$ are there?

$$P(n, n) = \frac{n!}{(n - n)!} = \frac{n!}{0!} = n!$$

How many ways are there of choosing $n$ out of $n$? 0 out of $n$?

$$\binom{n}{n} = \frac{n!}{n!0!} = 1$$

$$\binom{n}{0} = \frac{n!}{0!n!} = 1$$
More Questions

Q: How many ways are there of choosing $k$ things from $\{1, \ldots, n\}$ if 1 and 2 can’t both be chosen? (Suppose $n, k \geq 2$.)

A: First find all the ways of choosing $k$ things from $n$—$C(n, k)$. Then subtract the number of those ways in which both 1 and 2 are chosen:

- This amounts to choosing $k-2$ things from $\{3, \ldots, n\}$: $C(n - 2, k - 2)$.

Thus, the answer is

$$C(n, k) - C(n - 2, k - 2)$$

Q: What if order matters?

A: Have to compute how many ways there are of picking $k$ things, two of which are 1 and 2.

$$P(n, k) - k(k - 1)P(n - 2, k - 2)$$
**Q:** How many ways are there to distribute four distinct balls evenly between two distinct boxes (two balls go in each box)?

**A:** All you need to decide is which balls go in the first box.

\[ C(4, 2) = 6 \]

**Q:** What if the boxes are indistinguishable?

**A:** \( C(4, 2)/2 = 3 \).
Combinatorial Identities

There all lots of identities that you can form using $C(n, k)$. They seem mysterious at first, but there’s usually a good reason for them.

**Theorem 1:** If $0 \leq k \leq n$, then

$$C(n, k) = C(n, n - k).$$

**Proof:**

$$C(n, k) = \frac{n!}{k!(n - k)!} = \frac{n!}{(n - k)!(n - (n - k))!} = C(n, n - k)$$

**Q:** Why should choosing $k$ things out of $n$ be the same as choosing $n - k$ things out of $n$?

**A:** There’s a 1-1 correspondence. For every way of choosing $k$ things out of $n$, look at the things not chosen: that’s a way of choosing $n - k$ things out of $n$.

This is a better way of thinking about Theorem 1 than the combinatorial proof.
**Theorem 2:** If $0 < k < n$ then

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

**Proof 1:** (Combinatorial) Suppose we want to choose $k$ objects out of \{1, \ldots, n\}. Either we choose the last one ($n$) or we don’t.

1. How many ways are there of choosing $k$ without choosing the last one? $C(n-1, k)$.

2. How many ways are there of choosing $k$ including $n$? This means choosing $k - 1$ out of \{1, \ldots, n - 1\}: $C(n-1, k-1)$.

**Proof 2:** Algebraic . . .

**Note:** If we define $C(n, k) = 0$ for $k > n$ and $k < 0$, Theorems 1 and 2 still hold.
Pascal’s Triangle

Starting with $n = 0$, the $n$th row has $n + 1$ elements:

$C(n, 0), \ldots, C(n, n)$

Note how Pascal’s Triangle illustrates Theorems 1 and 2.
Theorem 3: For all \( n \geq 0 \):

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

Proof 1: \( \binom{n}{k} \) tells you all the way of choosing a subset of size \( k \) from a set of size \( n \). This means that the LHS is all the ways of choosing a subset from a set of size \( n \). The product rule says that this is \( 2^n \).

Proof 2: By induction. Let \( P(n) \) be the statement of the theorem.

Basis: \( \sum_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0 \). Thus \( P(0) \) is true.

Inductive step: How do we express \( \sum_{k=0}^{n} \binom{n}{k} \) in terms of \( n - 1 \), so that we can apply the inductive hypothesis?

- Use Theorem 2!
More combinatorial identities

**Theorem 4:** For any nonnegative integer \( n \)

\[
\sum_{k=0}^{n} k \binom{n}{k} = n 2^{n-1}
\]

**Proof 1:**

\[
\sum_{k=0}^{n} k \binom{n}{k} = \sum_{k=1}^{n} k \frac{n!}{(n-k)!k!}
\]

\[
= \sum_{k=1}^{n} n \frac{n!}{(n-k)!(k-1)!}
\]

\[
= n \sum_{k=1}^{n} \frac{(n-1)!}{(n-k)!(k-1)!}
\]

\[
= n \sum_{j=0}^{n-1} \frac{(n-1)!}{(n-1-j)!j!} \quad \text{[Let } j = k - 1 \text{]}
\]

\[
= n \sum_{j=0}^{n-1} \binom{n-1}{j}
\]

\[
= n 2^{n-1}
\]

**Proof 2:** LHS tells you all the ways of picking a subset of \( k \) elements out of \( n \) (a subcommittee) and designating one of its members as special (subcommittee chairman).

What’s another way of doing this? Pick the chairman first, and then the rest of the subcommittee!
Theorem 5:

\[(n - k)\binom{n}{k} = (k + 1)\binom{n}{k+1} = n\binom{n-1}{k}\]

Theorem 6:

\[C(n, k)C(n - k, j) = C(n, j)C(n - j, k) = C(n, k + j)C(k + j, j)\]

Theorem 7: \(P(n, k) = nP(n - 1, k - 1)\).
The Binomial Theorem

We want to compute \((x + y)^n\).

Some examples:

\[(x + y)^0 = 1\]
\[(x + y)^1 = x + y\]
\[(x + y)^2 = x^2 + 2xy + y^2\]
\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\]
\[(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]

The pattern of the coefficients is just like that in the corresponding row of Pascal’s triangle!

**Binomial Theorem:**

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k\]

**Proof 1:** By induction on \(n\). \(P(n)\) is the statement of the theorem.

**Basis:** \(P(1)\) is obviously OK. (So is \(P(0)\).)
Inductive step:

\[(x + y)^{n+1}\]
\[= (x + y)(x + y)^n\]
\[= (x + y)\sum_{k=0}^{n} \binom{n}{k}x^{n-k}y^k\]
\[= \sum_{k=0}^{n} \binom{n}{k}x^{n-k+1}y^k + \sum_{k=0}^{n} \binom{n}{k}x^{n-k}y^{k+1}\]
\[= \ldots \text{[Lots of missing steps]}\]
\[= y^{n+1} + \sum_{k=0}^{n} \left( \binom{n}{k} + \binom{n}{k-1} \right)x^{n-k+1}y^k\]
\[= y^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k}x^{n+1-k}y^k\]
\[= \sum_{k=0}^{n+1} \binom{n+1}{k}x^{n+1-k}y^k\]

**Proof 2:** What is the coefficient of the \(x^{n-k}y^k\) term in \((x + y)^n\)?
Using the Binomial Theorem

Q: What is \((x + 2)^4\)?

A:
\[
(x + 2)^4 = x^4 + C(4, 1)x^32 + C(4, 2)x^22^2 + C(4, 3)x2^3 + 2^4
= x^4 + 8x^3 + 24x^2 + 32x + 16
\]

Q: What is \((1.02)^7\) to 4 decimal places?

A:
\[
(1 + .02)^7 = 1^7 + C(7, 1)1^6(.02) + C(7, 2)1^5(.0004) + C(7, 3)(.000008) + \cdots
= 1 + .14 + .0084 + .0028 + \cdots
\approx 1.14868
\approx 1.1487
\]

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.
In the book they talk about the *multinomial theorem*. That’s for dealing with \((x + y + z)^n\).

They also talk about the *binomial series theorem*. That’s for dealing with \((x + y)^\alpha\), when \(\alpha\) is any *real* number (like 0.3).

You’re not responsible for these results.