Questions/Complaints About Homework?

Here's the procedure for homework questions/complaints:

1. Read the solutions first.
2. Talk to the person who graded it (check initials)
3. If (1) and (2) don't work, talk to me.

Further comments:

- There's no statute of limitations on grade changes
  - although asking questions right away is a good strategy
- Remember that 10/12 homeworks count. Each one is roughly worth 50 points, and homework is 35% of your final grade.
  - 16 homework points = 1% on your final grade
- Remember we're grading about 100 homeworks and graders are not expected to be mind readers. It's **your** problem to write clearly.
- Don't forget to staple your homework pages together, add the cover sheet, and put your name on clearly.
  - I'll deduct 2 points if that's not the case

Algorithmic number theory

Number theory used to be viewed as the purest branch of pure mathematics.

- Now it's the basis for most modern cryptography.
- Absolutely critical for e-commerce
  - How do you know your credit card number is safe?

Goal:

- To give you a basic understanding of the mathematics behind the RSA cryptosystem
  - Need to understand how prime numbers work

Division

For $a, b \in Z$, $a \neq 0$, a *divides* b if there is some $c \in Z$ such that $b = ac$.

- Notation: $a \mid b$
- Examples: $3 \mid 9$, $3 \not\mid 7$

If $a \mid b$, then a is a *factor* of b, b is a *multiple* of a.

Theorem 1: If $a, b, c \in Z$, then

1. if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.
2. If $a \mid b$ then $a \mid (bc)$
3. If $a \mid b$ and $b \mid c$ then $a \mid c$ (divisibility is transitive).

Proof: How do you prove this? Use the definition!

- E.g., if $a \mid b$ and $a \mid c$, then, for some d_1 and d_2,
$$b = ad_1 \text{ and } c = ad_2.$$
- That means $b + c = a(d_1 + d_2)$
- So $a \mid (b + c)$.

Other parts: homework.

Corollary 1: If $a \mid b$ and $a \mid c$, then $a \mid (mb + nc)$ for any integers m and n.

The division algorithm

Theorem 2: For $a \in Z$ and $d \in N$, $d > 0$, there exist unique $q, r \in Z$ such that $a = q \cdot d + r$ and $0 \leq r < d$.

- r is the remainder when a is divided by d

Notation: $r \equiv a \pmod{d}$; $a \bmod d = r$

Examples:

- Dividing 101 by 11 gives a quotient of 9 and a remainder of 2 ($101 \equiv 2 \pmod{11}$; $101 \bmod 11 = 2$).
- Dividing 18 by 6 gives a quotient of 3 and a remainder of 0 ($18 \equiv 0 \pmod{6}$; $18 \bmod 6 = 0$).

Proof: Let $q = \lfloor a/d \rfloor$ and define $r = a - q \cdot d$.

- So $a = q \cdot d + r$ with $q \in Z$ and $0 \leq r < d$ (since $q \cdot d \leq a$).

But why are q and d unique?

- Suppose $q \cdot d + r = q' \cdot d + r'$ with $q', r' \in Z$ and $0 \leq r' < d$.
- Then $(q' - q)d = (r - r')$ with $-d < r - r' < d$.
- The lhs is divisible by d so $r = r'$ and we're done.

Primes

- If $p \in N$, $p > 1$ is *prime* if its only positive factors are 1 and p.
- $n \in N$ is *composite* if $n > 1$ and n is not prime.
  - If n is composite then $a \mid n$ for some $a \in N$ with $1 < a < n$
  - Can assume that $a \leq \sqrt{n}$.
    - **Proof:** By contradiction:
      Suppose $n = bc$, $b > \sqrt{n}$, $c > \sqrt{n}$. But then $bc > n$, a contradiction.

Primes: $2, 3, 5, 7, 11, 13, \ldots$
Composites: $4, 6, 8, 9, \ldots$

Primality testing

How can we tell if $n \in N$ is prime?

The naive approach: check if $k \mid n$ for every $1 < k < n$.

- But at least 10^{m-1} numbers are $\leq n$, if n has m digits
  - 1000 numbers less than 1000 (a 4-digit number)
  - 1,000,000 less than 1,000,000 (a 7-digit number)

So the algorithm is *exponential time*!

We can do a little better

- Skip the even numbers
- That saves a factor of 2 $\longrightarrow$ not good enough
- Try only primes (Sieve of Eratosthenes)
  - Still doesn't help much

We can do much better:

- There is a polynomial time *randomized* algorithm
  - We will discuss this when we talk about probability
- In 2002, Agarwal, Saxena, and Kayal gave a (non-probabilistic) polynomial time algorithm
  - Saxena and Kayal were undergrads in 2002!

The Fundamental Theorem of Arithmetic

Theorem 3: Every natural number $n > 1$ can be uniquely represented as a product of primes, written in nondecreasing size.

- Examples: $54 = 2 \cdot 3^3$, $100 = 2^2 \cdot 5^2$, $15 = 3 \cdot 5$.

Proving that that n can be written as a product of primes is easy (by strong induction):

- Base case: 2 is the product of primes (just 2)
- Inductive step: If $n > 2$ is prime, we are done. If not, $n = ab$.
  - Must have $a < n$, $b < n$.
  - By I.H., both a and b can be written as a product of primes
  - So n is product of primes

Proving uniqueness is harder.

- We'll do that in a few days . . .

An Algorithm for Prime Factorization

Fact: If a is the smallest number > 1 that divides n, then a is prime.

Proof: By contradiction. (Left to the reader.)

- A *multiset* is like a set, except repetitions are allowed
  - $\{\{2, 2, 3, 3, 5\}\}$ is a multiset, not a set

PF(n): A prime factorization procedure

Input: $n \in N^+$
Output: PFS - a multiset of n's prime factors
PFS := $\emptyset$
for $a = 2$ to $\lfloor \sqrt{n} \rfloor$ do
  if $a \mid n$ **then** PFS := PF(n/a) $\cup \{\{a\}\}$ **return** PFS
if PFS = $\emptyset$ **then** PFS := $\{\{n\}\}$ [n is prime]

Example:
$$\begin{aligned} \text{PF}(7007) &= \{\{7\}\} \cup \text{PF}(1001) \\ &= \{\{7,7\}\} \cup \text{PF}(143) \\ &= \{\{7,7,11\}\} \cup \text{PF}(13) \\ &= \{\{7,7,11,13\}\}. \end{aligned}$$

The Complexity of Factoring

Algorithm PF runs in exponential time:

- We're checking every number up to $\sqrt{n}$

Can we do better?

- We don't know.
- Modern-day cryptography implicitly depends on the fact that we can't!

How Many Primes Are There?

Theorem 4: [Euclid] There are infinitely many primes.

Proof: By contradiction.

- Suppose that there are only finitely many primes: $p_1, \ldots, p_n$.
- Consider $q = p_1 \times \cdots \times p_n + 1$
- Clearly $q > p_1, ..., p_n$, so it can't be prime.
- So q must have a prime factor, which must be one of $p_1, \ldots, p_n$ (since these are the only primes).
- Suppose it is p_i.
  - Then $p_i \mid q$ and $p_i \mid p_1 \times \cdots \times p_n$
  - So $p_i \mid (q - p_1 \times \cdots \times p_n)$; i.e., $p_i \mid 1$ (Corollary 1)
  - Contradiction!

Largest currently-known prime (as of 5/04):

- $2^{24036583} - 1$: 7235733 digits
- Check www.utm.edu/research/primes

Primes of the form $2^p - 1$ where p is prime are called *Mersenne primes*.

- Search for large primes focuses on Mersenne primes

The distribution of primes

There are quite a few primes out there:

- Roughly one in every $\log(n)$ numbers is prime

Formally: let $\pi(n)$ be the number of primes $\leq n$:

Prime Number Theorem: $\pi(n) \sim n/\log(n)$; that is,

$$\lim_{n \to \infty} \pi(n)/(n/\log(n)) = 1$$

Why is this important?

- Cryptosystems like RSA use a secret key that is the product of two large (100-digit) primes.
- How do you find two large primes?
  - Roughly one of every 100 100-digit numbers is prime
  - To find a 100-digit prime;
    - Keep choosing odd numbers at random
    - Check if they are prime (using fast randomized primality test)
    - Keep trying until you find one
    - Roughly 100 attempts should do it

(Some) Open Problems Involving Primes

- Are there infinitely many Mersenne primes?
- *Goldbach's Conjecture*: every even number greater than 2 is the sum of two primes.
  - E.g., $6 = 3 + 3$, $20 = 17 + 3$, $28 = 17 + 11$
  - This has been checked out to 6×10^{16} (as of 2003)
  - Every sufficiently large integer ($> 10^{43,000}$!) is the sum of four primes
- Two prime numbers that differ by two are *twin primes*
  - E.g.: (3,5), (5,7), (11,13), (17,19), (41,43)
  - also $4,648,619,711,505 \times 2^{60,000} \pm 1$!

  Are there infinitely many twin primes?

All these conjectures are believed to be true, but no one has proved them.

Greatest Common Divisor (gcd)

Definition: For $a \in Z$ let $D(a) = \{k \in N : k \mid a\}$

- $D(a) = \{\text{divisors of } a\}$.

Claim. $|D(a)| < \infty$ if (and only if) $a \neq 0$.

Proof: If $a \neq 0$ and $k \mid a$, then $0 < k < a$.

Definition: For $a, b \in Z$, $CD(a, b) = D(a) \cap D(b)$ is the set of common divisors of a, b.

Definition: The *greatest common divisor* of a and b is

$$\gcd(a, b) = \max(CD(a, b)).$$

Examples:

- $\gcd(6, 9) = 3$
- $\gcd(13, 100) = 1$
- $\gcd(6, 45) = 3$

Def. a and b are *relatively prime* if $\gcd(a, b) = 1$.

- **Example:** 4 and 9 are relatively prime.
- Two numbers are relatively prime iff they have no common prime factors.

Efficient computation of $\gcd(a, b)$ lies at the heart of commercial cryptography.

Least Common Multiple (lcm)

Definition: The *least common multiple* of $a, b \in N^+$, $\text{lcm}(a, b)$, is the smallest $n \in N^+$ such that $a \mid n$ and $b \mid n$.

- **Examples:** $\text{lcm}(4, 9) = 36$, $\text{lcm}(4, 10) = 20$.

Computing the GCD

There is a method for calculating the gcd that goes back to Euclid:

- **Recall:** if $n > m$ and q divides both n and m, then q divides $n - m$ and $n + m$.

Therefore $\gcd(n, m) = \gcd(m, n - m)$.

- Proof: Show that $CD(n, m) = CD(m, n - m)$; i.e. show that q divides both n and m iff q divides both m and $n - m$. (If q divides n and m, then q divides $n - m$ by the argument above. If q divides m and $n - m$, then q divides $m + (n - m) = n$.)
- This allows us to reduce the gcd computation to a simpler case.

We can do even better:

- $\gcd(n, m) = \gcd(m, n - m) = \gcd(m, n - 2m) = \ldots$
- keep going as long as $n - qm \geq 0$ — $\lfloor n/m \rfloor$ steps

Consider $\gcd(6, 45)$:

- $\lfloor 45/6 \rfloor = 7$; remainder is 3 ($45 \equiv 3 \pmod 6$)
- $\gcd(6, 45) = \gcd(6, 45 - 7 \times 6) = \gcd(6, 3) = 3$

We can keep this up this procedure to compute $\gcd(n_1, n_2)$:

- If $n_1 \geq n_2$, write n_1 as $q_1 n_2 + r_1$, where $0 \leq r_1 < n_2$
  - $q_1 = \lfloor n_1/n_2 \rfloor$
- $\gcd(n_1, n_2) = \gcd(r_1, n_2)$
- Now $r_1 < n_2$, so switch their roles:
- $n_2 = q_2 r_1 + r_2$, where $0 \leq r_2 < r_1$
- $\gcd(r_1, n_2) = \gcd(r_1, r_2)$
- Notice that $\max(n_1, n_2) > \max(r_1, n_2) > \max(r_1, r_2)$
- Keep going until we have a remainder of 0 (i.e., something of the form $\gcd(r_k, 0)$ or $(\gcd(0, r_k))$
  - This is bound to happen sooner or later

Euclid's Algorithm

Input m, n [m, n natural numbers, $m \geq n$]
  $num \leftarrow m$; $denom \leftarrow n$ [Initialize num and $denom$]
  repeat until $denom = 0$
    $q \leftarrow \lfloor num/denom \rfloor$
    $rem \leftarrow num - (q * denom)$ [$num \bmod denom = rem$]
    $num \leftarrow denom$ [New num]
    $denom \leftarrow rem$ [New $denom$; note $num \geq denom$]
  endrepeat
Output num [$num = \gcd(m, n)$]

Example: $\gcd(84, 33)$

Iteration 1: $num = 84$, $denom = 33$, $q = 2$, $rem = 18$
Iteration 2: $num = 33$, $denom = 18$, $q = 1$, $rem = 15$
Iteration 3: $num = 18$, $denom = 15$, $q = 1$, $rem = 3$
Iteration 4: $num = 15$, $denom = 3$, $q = 5$, $rem = 0$
Iteration 5: $num = 3$, $denom = 0 \Rightarrow \gcd(84, 33) = 3$

Euclid's Algorithm: Correctness

How do we know this works?

- We need to prove that
  (a) the algorithm terminates and
  (b) that it correctly computes the gcd

We prove (a) and (b) simultaneously by finding appropriate loop invariants and using induction:

- Notation: Let num_k and $denom_k$ be the values of num and $denom$ at the beginning of the kth iteration.

$P(k)$ has three parts:

(1) $0 < num_{k+1} + denom_{k+1} < num_k + denom_k$

(2) $0 \leq denom_k \leq num_k$.

(3) $\gcd(num_k, denom_k) = \gcd(m, n)$

- Termination follows from parts (1) and (2): if $num_k + denom_k$ decreases and $0 \leq denom_k \leq num_k$, then eventually $denom_k$ must hit 0.
- Correctness follows from part (3).
- The induction step is proved by looking at the details of the loop.

Euclid's Algorithm: Complexity

Input m, n [m, n natural numbers, $m \geq n$]
  $num \leftarrow m$; $denom \leftarrow n$ [Initialize num and $denom$]
  repeat until $denom = 0$
    $q \leftarrow \lfloor num/denom \rfloor$
    $rem \leftarrow num - (q * denom)$
    $num \leftarrow denom$ [New num]
    $denom \leftarrow rem$ [New $denom$; note $num \geq denom$]
  endrepeat
Output num [$num = \gcd(m, n)$]

How many times do we go through the loop in the Euclidean algorithm:

- Best case: Easy. Never!
- Average case: Too hard
- Worst case: Can't answer this exactly, but we can get a good upper bound.
  - See how fast $denom$ goes down in each iteration.

Claim: After two iterations, $denom$ is halved:

- Recall $num = q * denom + rem$. Use $denom'$ and $denom''$ to denote value of $denom$ after 1 and 2 iterations. Two cases:
  1. $rem \leq denom/2 \Rightarrow denom' \leq denom/2$ and $denom'' < denom/2$.
  2. $rem > denom/2$. But then $num' = denom$, $denom' = rem$. At next iteration, $q = 1$, and $denom'' = rem' = num' - denom' < denom/2$
- How long until $denom$ is ≤ 1?
  - $< 2\log_2(m)$ steps!
- After at most $2\log_2(m)$ steps, $denom = 0$.

The Extended Euclidean Algorithm

Theorem 5: For $a, b \in N$, not both 0, we can compute $s, t \in Z$ such that

$$\gcd(a, b) = sa + tb.$$

- **Example:** $\gcd(9, 4) = 1 = 1 \cdot 9 + (-2) \cdot 4$.

Proof: By strong induction on $\max(a, b)$. Suppose without loss of generality $a \leq b$.

- If $\max(a, b) = 1$, then must have $b = 1$, $\gcd(a, b) = 1$
  - $\gcd(a, b) = 0 \cdot a + 1 \cdot b$.
- If $\max(a, b) > 1$, there are three cases:
  - $a = 0$; then $\gcd(0, b) = b = 0 \cdot a + 1 \cdot b$
  - $a = b$; then $\gcd(a, b) = a = 1 \cdot a + 0 \cdot b$
  - If $0 < a < b$, then $\gcd(a, b) = \gcd(a, b - a)$. Moreover, $\max(a, b) > \max(a, b - a)$. Thus, by IH, we can compute s, t such that

    $$\gcd(a, b) = \gcd(a, b-a) = sa+t(b-a) = (s-t)a+tb.$$

Note: this computation basically follows the "recipe" of Euclid's algorithm.

Example of Extended Euclidean Algorithm

Recall that $\gcd(84, 33) = \gcd(33, 18) = \gcd(18, 15) = \gcd(15, 3) = \gcd(3, 0) = 3$

We work backwards to write 3 as a linear combination of 84 and 33:

$3 = 18 - 15$

[Now 3 is a linear combination of 18 and 15]

$= 18 - (33 - 18)$

$= 2(18) - 33$

[Now 3 is a linear combination of 18 and 33]

$= 2(84 - 2 \times 33)) - 33$

$= 2 \times 84 - 5 \times 33$

[Now 3 is a linear combination of 84 and 33]

Some Consequences

Corollary 2: If a and b are relatively prime, then there exist s and t such that $as + bt = 1$.

Corollary 3: If $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Proof:

- Exist $s, t \in Z$ such that $sa + tb = 1$
- Multiply both sides by c: $sac + tbc = c$
- Since $a \mid bc$, $a \mid sac + tbc$, so $a \mid c$

Corollary 4: If p is prime and $p \mid \Pi_{i=1}^n a_i$, then $p \mid a_i$ for some $1 \leq i \leq n$.

Proof: By induction on n:

- If $n = 1$: trivial.

Suppose the result holds for n and $p \mid \Pi_{i=1}^{n+1} a_i$.

- note that $p \mid \Pi_{i=1}^{n+1} a_i = (\Pi_{i=1}^n a_i)a_{n+1}$.
- If $p \mid a_{n+1}$ we are done.
- If not, $\gcd(p, a_{n+1}) = 1$.
- By Corollary 3, $p \mid \Pi_{i=1}^n a_i$
- By the IH, $p \mid a_i$ for some $1 \leq i \leq n$.

The Fundamental Theorem of Arithmetic, II

Theorem 3: Every $n > 1$ can be represented uniquely as a product of primes, written in nondecreasing size.

Proof: Still need to prove uniqueness. We do it by strong induction.

- Base case: Obvious if $n = 2$.

Inductive step. Suppose OK for $n' < n$.

- Suppose that $n = \Pi_{i=1}^s p_i = \Pi_{j=1}^r q_j$.
- $p_1 \mid \Pi_{j=1}^r q_j$, so by Corollary 4, $p_1 \mid q_j$ for some j.
- But then $p_1 = q_j$, since both p_1 and q_j are prime.
- But then $n/p_1 = p_2 \cdots p_s = q_1 \cdots q_{j-1}q_{j+1} \cdots q_r$
- Result now follows from I.H.