CS280, Spring 2004: Final

This exam is out of 70; the amount for each question is as marked. Clearly explain your reasoning. (Remember, we can't read your mind! Besides, it gives us a chance to give you partial credit.) Don't forget to put your name and student number on each blue book you use. You can answer the questions in any order, just as long as you mark clearly which question you're answering. A good strategy is to do the ones you find easiest first.

Exams should be available next Tuesday or Wednesday; your grades will be posted on the CMS then too. You can pick up your exam in 4146 Upson after they're graded. We'll send out a message when that's done.

Good luck!

1. [4 points] Which of the following relations on $\{0, 1, 2, 3\}$ is an equivalence relation. (If it is, explain why. If it isn't, explain why not.) Just saying "Yes" or "No" with no explanation gets 0 points.)

   (a) $\{(0, 0,), (1, 1), (2, 2), (3, 3)\}$

   (b) $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

2. [5 points] Recall that a composite n is a Carmichael number if for any b relatively prime to n, $b^{n-1} \equiv 1 \pmod{n}$. Show that 1105 is a Carmichael number. [Hint: $1105 = 5 \cdot 13 \cdot 17$.]

3. [4 points] Consider any six natural numbers $n_1, \ldots, n_6$. Show that the sum of some subsequence of consecutive numbers is divisible by 6 (e.g., perhaps $n_3+n_4+n_5$ is divisible by 6, or n_4 itself is divisible by 6, or $n_2 + n_3$ is divisible by 6). [Hint: Look at the sums 0, n_1, $n_1 + n_2$, $n_1 + n_2 + n_3$, $\ldots$, $n_1 + n_2 + \cdots + n_6$, and think in terms of mod 6.]

4. [5 points] What is the coefficient of x^{25} in the binomial expansion of $(2x - \frac{3}{x^2})^{58}$? (There's no need to simplify the expression.)

5. [5 points] Prove that $3^n \geq n^3$ for all $n \geq 3$.

6. [3 points] How many 5-card hands have exactly 3 kings?

7. [4 points] A committee of 7 is to be chosen from 8 men and 9 women. How many contain either Alice or Bob, but not both? [You do *not* have to simplify the expression that you get.]

8. [5 points] There are N different types of coupons. Each time a coupon collector obtains a coupon it is equally likely to be any one of those. What is the probability that the collector needs more than k ($k > N$) coupons to complete his collection (have at least one of each type of coupon)? (You do *not* have to simplify the expression that you get.) [Hint: In other words, what is the probability that at least one type coupon is missing among the first k coupons.]

9. [10 points] For each of the following clearly answer yes or no (no justification is needed nor will it help you). If you answer correctly you get +1 but if you give the wrong answer you get -1 points. A skipped item earns 0 points. In any case the overall grade for this particular problem will not be less than zero.

   (a) If A and B are disjoint events then they are independent.

   (b) If A and B are independent events, then so are $\overline{A}$ and B.

   (c) A sum of Bernoulli random variables is a binomial random variable.

   (d) If X and Y are independent then $V(X+Y) = V(X) + V(Y)$.

   (e) If a is a real number and X is a random variable, then $V(aX) = aV(X)$.

   (f) If X and Y are *not* independent then $E(X+Y) \neq E(X) + E(Y)$.

   (g) If A and B are independent events then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

   (h) Suppose that you have a coin that has probability .2 of landing heads, and you toss it 100 times. Let X be the number of times that the coin lands heads. Then $Pr(X \geq 60) \leq 1/100$.

   (i) Let T_n be the number of times a fair coin lands heads after being flipped n times. Then
$$\lim_{n\to\infty} \Pr\left(\left|\frac{T_n}{n} - \frac{1}{2}\right| < .01\right) = 1.$$

   (j) Taking T_n as above,
$$\lim_{n\to\infty} \Pr\left(\left|\frac{2T_n - n}{\sqrt{n}}\right| < .01\right) = 1.$$

10. [4 points] There are three cards. The first is red on both sides; the second is black on both sides; and the third is red on one side and black on the other side. A card is randomly selected and randomly placed on the table. The color that we see is red. What is the probability that the hidden side is black?

11. [4 points] Two fair dice are rolled. What is the probability at least one lands on 6 given that the dice land on different numbers?

12. [5 points]

    Input: n (a positive integer)
    $\mathit{factorial} \leftarrow 1$
    $i \leftarrow 1$
    while $i < n$ **do**
    $\quad i \leftarrow i + 1$
    $\quad \mathit{factorial} \leftarrow i * \mathit{factorial}$

    Prove that the program terminates with $\mathit{factorial} = n!$ given input n, using an appropriate loop invariant.

13. [7 points] Translate the following argument into propositional logic, then determine whether it is valid. (Remember that an argument is valid if, whenever the premises are true, the conclusion is true.)

    If I like mathematics, then I will study.
    Either I don't study or I pass mathematics.
    If I don't graduate, then I didn't pass mathematics.

    If I like mathematics, then I will graduate

    [3 points for the translation, and 4 points for proving that it is valid or not valid.]

14. [5 points] Suppose that $K(x, y)$ "x knows (i.e., is acquainted with) y". Translate each of the following sentences into first-order logic:

    (a) Nobody knows Alice.

    (b) Sam knows everyone.

    (c) There is someone that Sam doesn't know.

    (d) Everyone knows someone.

    (e) Sam knows everyone that David knows.