| Discrete Structures | Problem Set 6 |
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| CS 280 Spring 2005 | Due Wednesday, March 16, 2005 |

Reading: Rosen Sections 1.4-1.6 and 4.1-4.3

**(1)** For sets $A$, $B$ and $C$, are the following equations true or false? Prove that your answer is correct.

(a) $(A - C) \cup (B - C) = (A \cup B) - C$

(b) $(A - B) \cup (B - C) = (A \cup B) - C$

**(2)** For a given $n$, what value of $k$ maximizes $\binom{n}{k}$? Prove that your answer is correct.

**(3)** Prove the following equation is true for all $0 \leq n$ :

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}.$$

**(4)** Let $S$ be a set of $n$ elements and consider a collection $\mathcal{A}$ of pairwise disjoint non-empty subsets of $S$. What is maximum cardinality $\mathcal{A}$ can have? Prove your answer is correct.

**(5)** Consider customer histories on Amazon. Suppose books are classified into $n$ genres (like biography, history, science fiction, etc). We are interested in sequences (or histories) of $k$ purchases. We say that a sequence is highly focused, if all but at most one of the $k$ purchases come from one genre. Let $P(n, k)$ denote the number of different purchase histories of length $k$ from the $n$ genres. (So, for $k = 2$, one possible history would be "science fiction, biography"). And let $F(n, k)$ be the number of highly focused histories. Express the ratio $F(n, k)/P(n, k)$ as a function of $n$ and $k$. Briefly explain why your formula is correct.

**(6)** In using distributed computing, it is often useful to keep each information on multiple processors, so that if some of them stops working the information is still accessible. However, having to update the information everywhere can also be a problem. One way to organize information is to use a quorum system. Let $S$ be the set of processors. We say that a collection $\mathcal{Q}$ of subsets of $S$ forms a quorum system if any two sets $A, B \in \mathcal{Q}$ intersect. The idea is that if the latest information is made available on a set $A \in \mathcal{Q}$ in the quorum system, and then later all elements of another set $B \in \mathcal{Q}$ are querried about the information, then the latest information will be obtained from the machine in $A \cap B$. (We'll need to also store the time of the last update so the latest version is recognized easily.) Let $q(n)$ be the maximum possible size of a quorum system on an $n$ element set. Give a formula for $q(n)$, and prove that your formula is correct.

**(7 optional)** In understanding sequences (such as DNA sequences of spicies), one is often looking for special properties of given strings. Here we will consider strings of 0s and 1s of a given length $n$, such as the string 0001000101010111 of length $n = 16$. An example of a special property is if the string has exactly $k$ occurences of 01 for some $k$. The example string above has this property with $k = 5$. To understand how important such a property is, we may want to know what is the percentage of strings of length $n$ that have this property. More precisely, let $p(n)$ be the number of 0/1 strings of length $n$, and let $s(n, k)$ be the number of 0/1 strings of length $n$ that have exactly $k$ occurences of 01. Give a formula for $p(n)$, $s(n, k)$ and $s(n, 2)/p(n)$ (as a function of $n$ and $k$).