Discrete Structures	**Problem Set 1**
CS 280 Spring 2005	**Due February 2, 2005**

Reading: Rosen Section 1.1-1.3 and Section 3.1. Please read the **Homework Information** on the Web about information on grading criteria, late homework policy, etc. Also note: homeworks count for 30% of the grade, and **no homework will be dropped.** For example of style of writing proofs see examples in Section 3.1.

(1) Construct truth tables for the following Boolean expressions (as in Section 1.1 of Rosen's book).

(a) $(p \wedge q) \rightarrow \neg p$

(b) $(p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$

(2) We talked about logical connectors like $\vee$ $\wedge$, $\neg$. We showed that a few others, like the exclusive $\oplus$, and a few others can be expressed out of the basic ones $\vee$, $\wedge$ and $\neg$. In fact, one can even express $\wedge$ using $\vee$ and $\neg$. For two statements p and q, write $p \wedge q$ as an expression using only $\vee$ and $\neg$.

(3) Boolean formulas are extensively used for designing classification rules in Artificial Intelligence and data mining. For example, a direct mailing advertiser for the Cornell Campus store, would like to send his catalog to all current and past Cornell students. It would be too expensive to mail to all of them, so they come up with the following rules.

- **Rule 1:** (student x, who lived in one of the freshmen dorms during the years 1998-2004) AND (x was a Art or Hotel student)
- **Rule 2:** (student x, who lived in one of the freshmen dorms during the years 1995-2004) AND (x was a Art or Hotel student or a Computer Science Major from Engineering)
- **Rule 3:** (student x, who lived in one of the freshmen dorms during the years 1998-2004) AND (x was a Computer Science Major)

(a) Give an example of a person x who satisfied Rule 1 but does not satisfy either Rule 2 or Rule 3. If no such person is possible, explain why not.

(b) Give an example of a person x who satisfied Rule 2 but does not satisfy either Rule 1 or Rule 3. If no such person is possible, explain why not.

(c) Give an example of a person x who satisfied Rule 3 but does not satisfy either Rule 1 or Rule 2. If no such person is possible, explain why not.

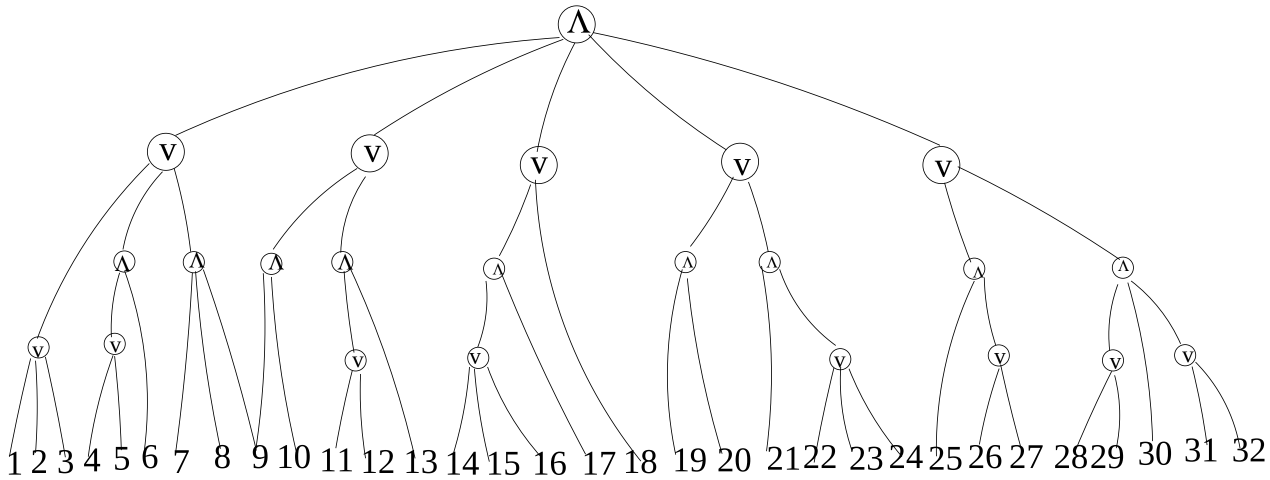

Figure 1: Monotone Formula for Problem 5.

(4) Systems often use logical expressions as operation rules. For example, a system consisting of n machines may be in acceptable state if at least one of the machines is functional. It is useful to express such statements via logical expression out of basic statements. For example, if we use $P(x)$ to mean that "Machine x is functional", than the above statement can be stated as $\exists x P(x)$.

Wanting to be a bit more secure, they want to switch to requiring that at least two machines are functional. Write this statement using the mathematical formalism $\exists$, $\forall$, etc, and the basic $P(x)$ statement above.

(5) Recall from lecture that a *monotone circuit* or *monotone Boolean formula* is a circuit or formula built using only $\wedge$, $\vee$ and variables (no negations). See the figure below for an example. We defined *positive coalition* (a set of variables such that setting this set of variables true, guarantees that the formula is true, independent of the setting of all other variables), and a *negative coalition* (a set of variables such that setting these variables false, guarantees that the formula is false, independent of the setting of all other variables).

Consider the monotone Boolean formula depicted by the circuit on Figure 1. It has 32 variables labelled $1, \ldots, 32$. Give a smallest positive coalition for this formula. Is this smallest positive coalition unique? You do not have to prove why your answer is correct.

(Note: there are far too many variables to try all subsets explicitly. You need think about how a small subset can be a positive coalition.)

(6) Consider the following claim.

Claim: If a monotone Boolean Formula has at least two variables, and its top gate is a $\wedge$, then a positive coalition has at least two variables.

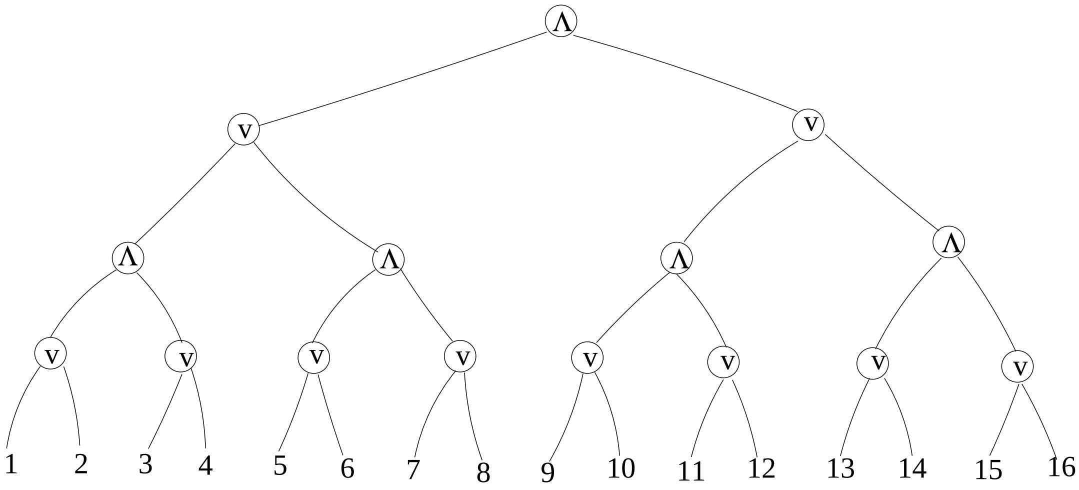

Figure 2: Monotone Formula for Problem 7.

Is this true or false? If you believe it is true, prove the claim. If you believe it is false, disprove it, by giving a formula that satisfied all the condition and yet has a positive coalition of size 1.

(7) We say that a variable x is *influential* if there is a setting of all variables, so that the formula is true, and if we change x's value without changing any other assignment the formula becomes false. Consider the formula depicted on Figure 2 on 16 variables labelled $1, \ldots, 16$, decide which of the 16 variables is *influential* (if any), and prove your claim.

(8 optional) Consider the following definition of a monotone formula. There are n variables. The formula is built by levels. On the first level we take the all pairs of variables (x, y). Generally, on every level, we consider all pairs of values computed in the previous level. If this is an odd numbered level, we take $\vee$ of the pairs, if this is an even level, we take $\wedge$. Finally, at the last level we take a single $\vee$ or $\wedge$ of the values computed at the previous level. Again if the last level is an odd numbered level we take $\vee$, if it is an even numbered level we take $\wedge$. Suppose there are k levels. For an example of such a formula for $n = 4$ variables, and $k = 2$ levels, see Figure3. Give the size of the smallest positive and negative coalition for all values of n and k, and explain your answer.

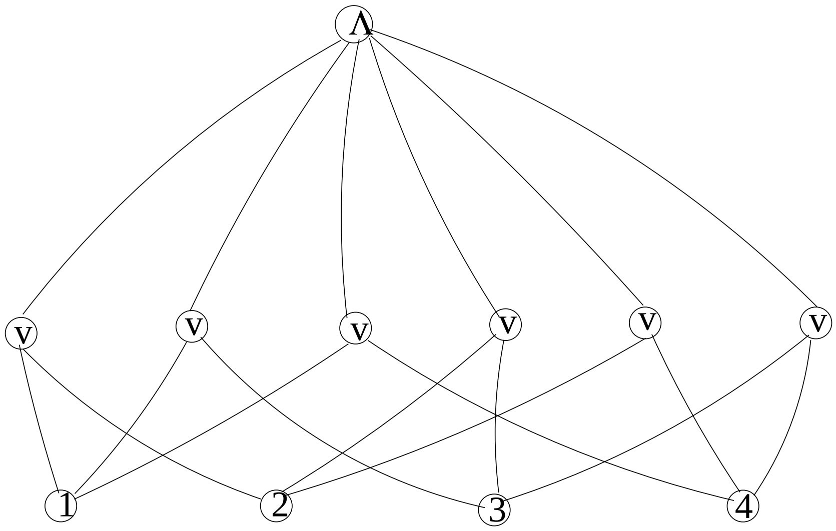

Figure 3: Monotone Formula for Problem 8.