Total Probability Theorem

• **Claim.** If $B \subset A$ then $\Pr(B) \leq \Pr(A)$.

• **Proof.** $A = B \cup (A \setminus B)$, so

  $$\Pr(A) = \Pr(B) + \Pr(A \setminus B) \geq \Pr(B).$$

• **Def.** The events $A_1, \ldots, A_n$ form a partition of the sample space $\Omega$ if

  1. $A_i$ are mutually exclusive: $A_i \cap A_j = \emptyset$ for $i \neq j$.
  2. $A_1 \cup \ldots \cup A_n = \Omega$.

• **Total Probability Theorem.** Let $A_1, \ldots, A_n$ be a partition of $\Omega$. For any event $B$,

  $$\Pr(B) = \sum_{j=1}^{n} \Pr(A_j) \Pr(B|A_j).$$

• **Proof.** $B = \bigcup (B \cap A_j)$ (disjoint union), so

  $$\Pr(B) = \sum_{j=1}^{n} \Pr(B \cap A_j).$$

  The theorem follows from $\Pr(B \cap A_j) = \Pr(A_j) \Pr(B|A_j)$.

• The latter holds for $A_j$ with $\Pr(A_j) = 0$ if we define $\Pr(A_j) \Pr(B|A_j) := 0$ since then $P(B \cap A_j) = 0$. 

Example

• In a certain county
  - 60% of registered voters are Republicans
  - 30% are Democrats
  - 10% are Independents.

• When those voters were asked about increasing military spending
  - 40% of Republicans opposed it
  - 65% of the Democrats opposed it
  - 55% of the Independents opposed it.

• What is the probability that a randomly selected voter in this county opposes increased military spending?
\begin{itemize}
    \item \(\Omega = \{\text{registered voters in the county}\}\)
    \item \(R = \{\text{registered republicans}\}, \Pr(R) = 0.6\)
    \item \(D = \{\text{registered democrats}\}, \Pr(D) = 0.3\)
    \item \(I = \{\text{registered independents}\}, \Pr(I) = 0.1\)
    \item \(B = \{\text{registered voters opposing increased military spending}\}\)
    \item \(\Pr(B|R) = 0.4, \Pr(B|D) = 0.65, \Pr(B|I) = 0.55.\)
\end{itemize}

By the total probability theorem:

\[
\Pr(B) = \Pr(B|R) \Pr(R) + \Pr(B|D) \Pr(D) + \Pr(B|I) \Pr(I)
= (0.4 \cdot 0.6) + (0.65 \cdot 0.3) + (0.55 \cdot 0.1) = 0.49.
\]
Bayes’ Theorem

• **Bayes Theorem.** Let $A_1, \ldots, A_n$ be a partition of $\Omega$. For any event $B$

$$\Pr(A_i|B) = \frac{\Pr(A_i) \Pr(B|A_i)}{\sum_{j=1}^{n} \Pr(A_j) \Pr(B|A_j)}.$$

• **Proof.**

$$\Pr(A_i|B) = \frac{\Pr(A_i \cap B)}{\Pr(B)} = \frac{\Pr(A_i) \Pr(B|A_i)}{\sum_{j=1}^{n} \Pr(A_j) \Pr(B|A_j)}.$$

• **Example.**

  • A registered voter from our county writes a letter to the local paper, arguing against increased military spending. What is the probability that this voter is a Democrat?
  • Presumably that is $\Pr(D|B)$, so by Bayes’ theorem:

$$\Pr(D|B) = \frac{0.65 \cdot 0.3}{(0.4 \cdot 0.6) + (0.65 \cdot 0.3) + (0.55 \cdot 0.1)} = \frac{0.195}{0.49} \approx 0.398.$$
AIDS

• Just for the heck of it Bob decides to take a test for AIDS and it comes back positive.

• The test is 99% effective (1% FP and FN).

• Suppose 0.3% of the population in Bob’s “bracket” has AIDS.

• What is the probability that he has AIDS?

• \[ \Omega = \{ \text{all the people in Bob’s bracket} \} \]

  • \[ A_1 = \{ \text{people in } \Omega \text{ with AIDS} \}, \Pr(A_1) = 0.003 \]
  • \[ A_2 = \{ \text{people in } \Omega \text{ without AIDS} \}, \Pr(A_2) = 0.997 \]
  • \[ B = \{ \text{people in } \Omega \text{ who would test positive} \} \]

  • \[ \Pr(B|A_1) = .99 \text{ and } \Pr(B|A_2) = .01 \]

  • By Bayes’ rule

\[
\Pr(A_1|B) = \frac{0.99 \cdot 0.003}{(0.99 \cdot 0.003) + (0.01 \cdot 0.997)} \\
\approx \frac{0.003}{0.003 + 0.01} \\
\approx 0.23.
\]
Random Variables

• What is the probability that:
  · in \( k \) out of \( n \) flips a coin will land on its head
  · \( k \) out of \( n \) randomly selected people will test positive to AIDS
  · \( k \) out of \( n \) rolled die will show an even number
  · a student will correctly guess \( k \) out of \( n \) multiple choice questions with a fixed number of choices.

• Mathematicians realized that often a problem in probability can be summarized in terms of a “random variable”.

• Definition. A random variable on a sample space \( \Omega \) is a function from \( \Omega \) to \( \mathbb{R} \).

• Example. A coin is flipped 10 times.
  · \( \Omega \) is the space of all sequences of \( H \) and \( T \) of length 10.
  · Let \( X \) count the number of heads.
  · For example \( X(\underbrace{HHTHHTTHTH}_{10}) = 6 \).

• Can you think of an analogous \( X \) for the other problems mentioned above?
• **Example.** What is the probability that a randomly chosen person in the class will weigh more than 160 lbs.?
  
  · A natural random variable in this case is the weight of the selected student.

• What is the difference between the range of values our two random variables can attain?
Probability Distributions

• There is a natural probabilistic structure induced on a random variable $X$ defined on $\Omega$:
  
  · The set $\{\omega \in \Omega : X(\omega) = c\}$ is an event.
  
  · So we can ask for
  
  \[ \Pr(X = c) = \Pr(\{\omega \in \Omega : X(\omega) = c\}). \]

**Example.** A biased coin ($\Pr(H) = 2/3$) is flipped twice.

  · Let $X$ count the number of heads:
    
    \[
    \begin{align*}
    \Pr(X = 0) &= \Pr(\{TT\}) = (1/3)^2 = 1/9. \\
    \Pr(X = 1) &= \Pr(\{HT, TH\}) = 2 \cdot 1/3 \cdot 2/3 = 4/9. \\
    \Pr(X = 2) &= \Pr(\{HH\}) = (2/3)^2 = 4/9.
    \end{align*}
    \]

• Similarly we might be interested in:

  \[ \Pr(X \leq c) = \Pr(\{\omega \in \Omega : X(\omega) \leq c\}), \]

  and more generally, for any $T \subset \mathbb{R}$:

  \[ \Pr(X \in T) = \Pr(\{\omega \in \Omega : X(\omega) \in T\}). \]

• In our coin example,

  \[ \Pr(X \leq 1) = \Pr(\{TT, HT, TH\}) = 1/9 + 4/9 = 5/9. \]