Pascal’s Triangle

Starting with \( n = 0 \), the \( n \)th row has \( n + 1 \) elements:

\[ C(n, 0), \ldots, C(n, n) \]

Note how Pascal’s Triangle illustrates Theorems 1 and 2.

**Theorem 3:** For all \( n \geq 0 \):

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

**Proof 1:** \( \binom{n}{k} \) tells you all the way of choosing a subset of size \( k \) from a set of size \( n \). This means that the LHS is *all* the ways of choosing a subset from a set of size \( n \). The product rule says that this is \( 2^n \).

**Proof 2:** By induction. Let \( P(n) \) be the statement of the theorem.

*Basis:* \( \sum_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0 \). Thus \( P(0) \) is true.

*Inductive step:* How do we express \( \sum_{k=0}^{n} C(n, k) \) in terms of \( n - 1 \), so that we can apply the inductive hypothesis?

- Use Theorem 2!
**Theorem 4:** For any nonnegative integer $n$

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

**Proof 1:**

\[
\begin{align*}
\Sigma_{k=0}^{n} k \binom{n}{k} &= \Sigma_{k=1}^{n} k \frac{n!}{(n-k)!k!} \\
&= \Sigma_{k=1}^{n} \frac{n!}{(n-k)!k!(k-1)!} \\
&= n \Sigma_{k=1}^{n} \frac{(n-1)!}{(n-k)!k!(k-1)!} \\
&= n \Sigma_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!k!} \\
&= n \Sigma_{k=0}^{n-1} \binom{n-1}{k} = n2^{n-1}
\end{align*}
\]

**Proof 2:** LHS tells you all the ways of picking a subset of $k$ elements out of $n$ (a subcommittee) and designating one of its members as special (subcommittee chairman).

What’s another way of doing this? Pick the chairman first, and then the rest of the subcommittee!
Theorem 5:

\[(n - k)\binom{n}{k} = (k + 1)\binom{n}{k + 1} = n\binom{n - 1}{k}\]

Theorem 6:

\[C(n, k)C(n - k, j) = C(n, j)C(n - j, k) = C(n, k + j)C(k + j, j)\]

Theorem 7: \(P(n, k) = nP(n - 1, k - 1)\).
The Binomial Theorem

We want to compute \((x + y)^n\).

Some examples:

\[
(x + y)^1 = x + y
\]
\[
(x + y)^2 = x^2 + 2xy + y^2
\]
\[
(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3
\]
\[
(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4
\]

The pattern of the coefficients is just like that in the corresponding row of Pascal’s triangle!

**Binomial Theorem:**

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

**Proof 1:** By induction on \(n\). \(P(n)\) is the statement of the theorem.

*Basis:* \(P(1)\) is obviously OK. (So is \(P(0)\).)
Inductive step:

\[
(x + y)^{n+1} = (x + y)(x + y)^n = (x + y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}
\]

... [Lots of missing steps]

\[
y^{n+1} + \sum_{k=0}^{n} \left( \binom{n}{k} + \binom{n}{k-1} \right) x^{n-k+1} y^k
\]

\[
y^{n+1} + \sum_{k=0}^{n} \binom{n+1}{k} x^{n+1-k} y^k
\]

\[
\sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k
\]

Proof 2: What is the coefficient of the \( x^{n-k} y^k \) term in \((x + y)^n\)?
Using the Binomial Theorem

Q: What is \((x + 2)^4\)?

A:
\[
(x + 2)^4 = x^4 + C(4, 1)x^3(2) + C(4, 2)x^22^2 + C(4, 3)x2^3 + 2^4
\]
\[
= x^4 + 8x^3 + 24x^2 + 32x + 16
\]

Q: What is \((1.02)^7\) to 4 decimal places?

A:
\[
(1 + .02)^7 = 1^7 + C(7, 1)1^6(.02) + C(7, 2)1^5(.0004) + C(7, 3)(.000008) + \cdots
\]
\[
= 1 + .14 + .0084 + .00028 + \cdots
\]
\[
\approx 1.14868
\]
\[
\approx 1.1487
\]

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.
In the book they talk about the *multinomial theorem*. That’s for dealing with \((x + y + z)^n\).

They also talk about the *binomial series theorem*. That’s for dealing with \((x + y)^\alpha\), when \(\alpha\) is any real number (like 0.3).

You’re not responsible for these results.