Pascal's Triangle

Starting with $n = 0$, the nth row has $n + 1$ elements:

$$C(n, 0), \ldots, C(n, n)$$

Note how Pascal's Triangle illustrates Theorems 1 and 2.

Theorem 3: For all $n \geq 0$:

$$\Sigma_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof 1: $\binom{n}{k}$ tells you all the way of choosing a subset of size k from a set of size n. This means that the LHS is *all* the ways of choosing a subset from a set of size n. The product rule says that this is 2^n.

Proof 2: By induction. Let $P(n)$ be the statement of the theorem.

Basis: $\Sigma_{k=0}^{0} \binom{0}{k} = \binom{0}{0} = 1 = 2^0$. Thus $P(0)$ is true.

Inductive step: How do we express $\Sigma_{k=0}^{n} C(n, k)$ in terms of $n - 1$, so that we can apply the inductive hypothesis?

- Use Theorem 2!

More Combinatorial Identities

Theorem 4: For any nonnegative integer n

$$\Sigma_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

Proof 1:

$$\begin{aligned}
& \Sigma_{k=0}^{n} k \binom{n}{k} \\
= \ & \Sigma_{k=1}^{n} k \tfrac{n!}{(n-k)!k!} \\
= \ & \Sigma_{k=1}^{n} \tfrac{n!}{(n-k)!(k-1)!} \\
= \ & n \Sigma_{k=1}^{n} \tfrac{(n-1)!}{(n-k)!(k-1)!} \\
= \ & n \Sigma_{k=0}^{n-1} \tfrac{(n-1)!}{(n-1-k)!k!} \\
= \ & n \Sigma_{k=0}^{n-1} C(n-1, k) \\
= \ & n 2^{n-1}
\end{aligned}$$

Proof 2: LHS tells you all the ways of picking a subset of k elements out of n (a subcommittee) and designating one of its members as special (subcomittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

Theorem 5:

$$(n - k) \binom{n}{k} = (k + 1) \binom{n}{(k + 1)} = n \binom{(n - 1)}{k}$$

Theorem 6:

$$\begin{aligned}
C(n, k)C(n - k, j) &= C(n, j)C(n - j, k) \\
&= C(n, k + j)C(k + j, j)
\end{aligned}$$

Theorem 7: $P(n, k) = nP(n - 1, k - 1)$.

The Binomial Theorem

We want to compute $(x+y)^n$.
Some examples:

$$(x+y)^1 = x+y$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

The pattern of the coefficients is just like that in the corresponding row of Pascal's triangle!

Binomial Theorem:

$$(x+y)^n = \Sigma_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

Proof 1: By induction on n. $P(n)$ is the statement of the theorem.

Basis: $P(1)$ is obviously OK. (So is $P(0)$.)

Inductive step:

$$\begin{aligned}
& (x+y)^{n+1} \\
= & \ (x+y)(x+y)^n \\
= & \ (x+y)\Sigma_{k=0}^{n}\tbinom{n}{k}x^{n-k}y^k \\
= & \ \Sigma_{k=0}^{n}\tbinom{n}{k}x^{n-k+1}y^k + \Sigma_{k=0}^{n}\tbinom{n}{k}x^{n-k}y^{k+1} \\
= & \ \ldots \qquad \text{[Lots of missing steps]} \\
= & \ y^{n+1} + \Sigma_{k=0}^{n}(\tbinom{n}{k}+\tbinom{n}{k-1})x^{n-k+1}y^k \\
= & \ y^{n+1} + \Sigma_{k=0}^{n}(\tbinom{n+1}{k})x^{n+1-k}y^k \\
= & \ \Sigma_{k=0}^{n+1}\tbinom{n+1}{k}x^{n+1-k}y^k
\end{aligned}$$

Proof 2: What is the coefficient of the $x^{n-k}y^k$ term in $(x+y)^n$?

Using the Binomial Theorem

Q: What is $(x+2)^4$?

A:

$$\begin{aligned}
& (x+2)^4 \\
= & \ x^4 + C(4,1)x^3(2) + C(4,2)x^2 2^2 + C(4,3)x2^3 + 2^4 \\
= & \ x^4 + 8x^3 + 24x^2 + 32x + 16
\end{aligned}$$

Q: What is $(1.02)^7$ to 4 decimal places?

A:

$$\begin{aligned}
& (1+.02)^7 \\
= & \ 1^7 + C(7,1)1^6(.02) + C(7,2)1^5(.0004) + C(7,3)(.000008) + \cdots \\
= & \ 1 + .14 + .0084 + .00028 + \cdots \\
\approx & \ 1.14868 \\
\approx & \ 1.1487
\end{aligned}$$

Note that we have to go to 5 decimal places to compute the answer to 4 decimal places.

In the book they talk about the *multinomial theorem*. That's for dealing with $(x+y+z)^n$.

They also talk about the *binomial series theorem*. That's for dealing with $(x+y)^\alpha$, when α is any *real* number (like 0.3).

You're not responsible for these results.

Balls and Urns

"Balls and urns" problems are paradigmatic. Many problems can be recast as balls and urns problems, once we figure out which are the balls and which are the urns.

How many ways are there of putting b balls into u urns?

- That depends whether the balls are distinguishable and whether the urns are distinguishable

How many ways are there of putting 5 balls into 2 urns?

- If both balls and urns are distinguishable: $2^5 = 32$
  - Choose the subset of balls that goes into the first urn
  - Alternatively, for each ball, decide which urn it goes in
  - This assumes that it's OK to have 0 balls in an urn.
- If urns are distinguishable but balls aren't: 6
  - Decide how many balls go into the first urn: 0, 1, ..., 5
- If balls are distinguishable but urns aren't: $2^5/2 = 16$
- If balls and urns are indistinguishable: $6/2 = 3$

What if we had 6 balls and 2 urns?

- If balls and urns are distinguishable: 2^6
- If urns are distinguishable and balls aren't: 7
- If balls are distinguishable but urns aren't:

$$2^6/2 = 2^5$$

- If balls and urns are indistinguishable: 4
  - It can't be 7/2, since that's not an integer
  - The problem is that if there are 3 balls in each urn, and you switch urns, then you get the same solution

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.

Distinguishable Urns

How many ways can b distinguishable balls be put into u distinguishable urns?

- By the product rule, this is u^b

How many ways can b indistinguishable balls be put into u distinguishable urns?

$C(u + b - 1, b)$

Reducing Problems to Balls and Urns

Q1: How many different configurations are there in Towers of Hanoi with n rings?

A: The urns are the poles, the balls are the rings. Both are distinguishable.

- 3^n

Q2: How many solutions are there to the equation $x + y + z = 65$, if x, y, z are nonnegative integers?

A: You have 65 indistinguishable balls, and want to put them into 3 distinguishable urns (x, y, z). Each way of doing so corresponds to one solution.

- $C(67, 65) = 67 \times 33 = 2211$

Q3: How many ways can 8 electrons be assigned to 4 energy states?

A: The electrons are the balls; they're indistinguishable. The energy states are the urns; they're distinguishable.

- $C(11, 8) = (11 \times 10 \times 9)/6 = 165$

Indistinguishable Urns

How many ways can b distinguishable balls be put into u indistinguishable urns?

First view the urns as distinguishable: u^b

For every solution, look at all $u!$ permutations of the urns. That should count as one solution.

- By the Division Rule, we get: $u^b/u!$?

This can't be right! It's not an integer (e.g. $7^3/7!$).

What's wrong?

The situation is even worse when we have indistinguishable balls in indistinguishable urns. (See the book.)