Modular Arithmetic

Remember: $a \equiv b \pmod{m}$ means a and b have the same remainder when divided by m.

- Equivalently: $a \equiv b \pmod{m}$ iff $m \mid (a - b)$
- a is *congruent* to b mod m

Theorem 7: If $a_1 \equiv a_2 \pmod{m}$ and $b_1 \equiv b_2 \pmod{m}$, then

(a) $(a_1 + b_1) \equiv (a_2 + b_2) \pmod{m}$

(b) $a_1 b_1 \equiv a_2 b_2 \pmod{m}$

Proof: Suppose

- $a_1 = c_1 m + r$, $a_2 = c_2 m + r$
- $b_1 = d_1 m + r'$, $b_2 = d_2 m + r'$

So

- $a_1 + b_1 = (c_1 + d_1)m + (r + r')$
- $a_2 + b_2 = (c_2 + d_2)m + (r + r')$

$m \mid ((a_1 + b_1) - (a_2 + b_2) = ((c_1 + d_1) - (c_2 + d_2))m$

- Conclusion: $a_1 + b_1 \equiv a_2 + b_2 \pmod{m}$.

For multiplication:

- $a_1 b_1 = (c_1 d_1 m + r' c_1 + r d_1)m + rr'$
- $a_2 b_2 = (c_2 d_2 m + r' c_2 + r d_2)m + rr'$

$m \mid (a_1 b_1 - a_2 b_2)$

- Conclusion: $a_1 b_1 \equiv a_2 b_2 \pmod{m}$.

Bottom line: addition and multiplication carry over to the modular world.

Modular arithmetic has lots of applications.

- Here are four ...

Hashing

Problem: How can we efficiently store, retrieve, and delete records from a large database?

- For example, students records.

Assume, each record has a unique key

- E.g. student ID, Social Security #

Do we keep an array sorted by the key?

- Easy retrieval but difficult insertion and deletion.

How about a table with an entry for every possible key?

- Often infeasible, almost always wasteful.
- There are 10^{10} possible social security numbers.

Solution: store the records in an array of size N, where N is somewhat bigger than the expected number of records.

- Store record with id k in location $h(k)$
  - h is the *hash function*
  - Basic hash function: $h(k) := k \pmod{N}$.
- A collision occurs when $h(k_1) = h(k_2)$ and $k_1 \neq k_2$.
  - Choose N sufficiently large to minimize collisions
- Lots of techniques for dealing with collisions

Pseudorandom Sequences

For randomized algorithms we need a random number generator.

- Most languages provide you with a function "rand".
- There is nothing random about rand!
  - It creates an apparently random sequence deterministically
  - These are called *pseudorandom sequences*

A standard technique for creating psuedorandom sequences: the *linear congruential method*.

- Choose a modulus $m \in N^+$,
- a multiplier $a \in \{2, 3, \ldots, m - 1\}$, and
- an increment $c \in Z_m = \{0, 1, \ldots, m - 1\}$.
- Choose a seed $x_0 \in Z_m$
  - Typically the time on some internal clock is used
- Compute $x_{n+1} = a x_n + c \pmod{m}$.

Warning: a poorly implemented rand, such as in C, can wreak havoc on Monte Carlo simulations.

ISBN Numbers

Since 1968, most published books have been assigned a 10-digit ISBN numbers:

- identifies country of publication, publisher, and book itself
- The ISBN number for DAM3 is 1-56881-166-7

All the information is encoded in the first 9 digits

- The 10th digit is used as a parity check
- If the digits are $a_1, \ldots, a_{10}$, then we must have
  $$a_1 + 2a_2 + \cdots + 9a_9 + 10a_{10} \equiv 0 \pmod{11}.$$
- For DAM3, get
  $$1 + 2 \times 5 + 3 \times 6 + 4 \times 8 + 5 \times 8 + 6 \times 1$$
  $$+7 \times 1 + 8 \times 6 + 9 \times 6 + 10 \times 7 = 286 \equiv 0 \pmod{11}$$
- This test always detects errors in single digits and transposition errors
  - Two arbitrary errors may cancel out

Similar parity checks are used in universal product codes (UPC codes/bar codes) that appear on almost all items

- The numbers are encoded by thicknesses of bars, to make them machine readable

Casting out 9s

Notice that a number is equivalent to the sum of its digits mod 9. This can be used as a way of checking your addition. [More in class]

Linear Congruences

The equation $ax = b$ for $a, b \in R$ is uniquely solvable if $a \neq 0$: $x = ba^{-1}$.

- Can we also (uniquely) solve $ax \equiv b \pmod{m}$?
- If x_0 is a solution, then so is $x_0 + km$ $\forall k \in Z$
  - ...since $km \equiv 0 \pmod{m}$.

So, uniqueness can only be mod m.

But even mod m, there can be more than one solution:

- Consider $2x \equiv 2 \pmod{4}$
- Clearly $x \equiv 1 \pmod{4}$ is one solution
- But so is $x \equiv 3 \pmod{4}$!

Theorem 8: If $\gcd(a, m) = 1$ then there is a unique solution (mod m) to $ax \equiv b \pmod{m}$.

Proof: Suppose $r, s \in Z$ both solve the equation:

- then $ar \equiv as \pmod{m}$, so $m \mid a(r - s)$
- Since $\gcd(a, m) = 1$, by Corollary 3, $m \mid (r - s)$
- But that means $r \equiv s \pmod{m}$

So if there's a solution at all, then it's unique mod m.

Solving Linear Congruences

But why is there a solution to $ax \equiv b \pmod{m}$?

Key idea: find a^{-1} mod m; then $x \equiv ba^{-1} \pmod{m}$

- By Corollary 2, since $\gcd(a, m) = 1$, there exist s, t such that
  $$as + mt = 1$$
- So $as \equiv 1 \pmod{m}$
- That means $s \equiv a^{-1} \pmod{m}$
- $x \equiv bs \pmod{m}$

The Chinese Remainder Theorem

Suppose we want to solve a system of linear congruences:

Example: Find x such that

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

Can we solve for x? Is the answer unique?

Definition: $m_1, \ldots, m_n$ are *pairwise relatively prime* if each pair m_i, m_j is relatively prime.

Theorem 9 (Chinese Remainder Theorem): Let $m_1, \ldots, m_n \in N^+$ be pairwise relatively prime. The system

$$x \equiv a_i \pmod{m_i} \qquad i = 1, 2 \ldots n \qquad (1)$$

has a unique solution modulo $M = \Pi_1^n m_i$.

- The best we can hope for is uniqueness modulo M:
  - If x is a solution then so is $x + kM$ for any $k \in Z$.

Proof: First I show that there is a solution; then I'll show it's unique.

CRT: Existence

Key idea for existence:
Suppose we can find $y_1, \ldots, y_n$ such that

$$y_i \equiv a_i \pmod{m_i}$$
$$y_i \equiv 0 \pmod{m_j} \quad \text{if } j \neq i.$$

Now consider $y := \Sigma_{j=1}^n y_j$.

$$\Sigma_{j=1}^n y_j \equiv a_i \pmod{m_i}$$

- Since $y_i = a_i \bmod m_i$ and $y_j = 0 \bmod m_j$ if $j \neq i$.

So y is a solution!

- Now we need to find $y_1, \ldots, y_n$.
- Let $M_i = M/m_i = m_1 \times \cdots \times m_{i-1} \times m_{i+1} \times \cdots \times m_n$.
- $\gcd(M_i, m_i) = 1$, since m_j's pairwise relatively prime
  - No common prime factors among any of the m_j's

  Choose y_i' such that $(M_i)y_i' \equiv a_i \pmod{m_i}$
  - Can do that by Theorem 8, since $\gcd(M_i, m_i) = 1$.

  Let $y_i = y_i' M_i$.
  - y_i is a multiple of m_j if $j \neq i$, so $y_i \equiv 0 \pmod{m_j}$
  - $y_i = y_i' M_i \equiv a_i \pmod{m_i}$ by construction.

So $y_1 + \cdots + y_n$ is a solution to the system, mod M.

CRT: Example

Find x such that

$$x \equiv 2 \pmod 3$$
$$x \equiv 3 \pmod 5$$
$$x \equiv 2 \pmod 7$$

Find y_1 such that $y_1 \equiv 2 \pmod 3$, $y_1 \equiv 0 \pmod{5/7}$:

- y_1 has the form $y_1' \times 5 \times 7$
- $35y_1' \equiv 2 \pmod 3$
- $y_1' = 1$, so $y_1 = 35$.

Find y_2 such that $y_2 \equiv 3 \pmod 5$, $y_2 \equiv 0 \pmod{3/7}$:

- y_2 has the form $y_2' \times 3 \times 7$
- $21y_2' \equiv 3 \pmod 5$
- $y_2' = 3$, so $y_2 = 63$.

Find y_3 such that $y_3 \equiv 2 \pmod 7$, $y_3 \equiv 0 \pmod{3/5}$:

- y_3 has the form $y_3' \times 3 \times 5$
- $15y_3' \equiv 2 \pmod 7$
- $y_3' = 2$, so $y_3 = 30$.

Solution is $x = y_1 + y_2 + y_3 = 35 + 63 + 30 = 128$

CRT: Uniqueness

What if x, y are both solutions to the equations?

- $x \equiv y \pmod{m_i} \Rightarrow m_i \mid (x - y)$, for $i = 1, \ldots, n$
- **Claim:** $M = m_1 \cdots m_n \mid (x - y)$
- so $x \equiv y \pmod M$

Theorem 10: If $m_1, \ldots, m_n$ are pairwise relatively prime and $m_i \mid b$ for $i = 1, \ldots, n$, then $m_1 \cdots m_n \mid b$.

Proof: By induction on n.

- For $n = 1$ the statement is trivial.

Suppose statement holds for $n = N$.

- Suppose $m_1, \ldots, m_{N+1}$ relatively prime, $m_i \mid b$ for $i = 1, \ldots, N + 1$.
- by IH, $m_1 \cdots m_N \mid b \Rightarrow b = m_1 \cdots m_N c$ for some c
- By assumption, $m_{N+1} \mid b$, so $m \mid (m_1 \cdots m_N)c$
- $\gcd(m_1 \cdots m_N, m_{N+1}) = 1$ (since m_i's pairwise relatively prime $\Rightarrow$ no common factors)
- by Corollary 3, $m_{N+1} \mid c$
- so $c = dm_{N+1}$, $b = m_1 \cdots m_N m_{N+1} d$
- so $m_1 \cdots m_{N+1} \mid b$.

An Application of CRT: Computer Arithmetic with Large Integers

Suppose we want to perform arithmetic operations (addition, multiplication) with extremely large integers

- too large to be represented easily in a computer

Idea:

- Step 1: Find suitable moduli $m_1, \ldots, m_n$ so that m_i's are relatively prime and $m_1 \cdots m_n$ is bigger than the answer.
- Step 2: Perform all the operations mod m_j, $j = 1, \ldots, n$.
  - This means we're working with much smaller numbers (no bigger than m_j)
  - The operations are much faster
  - Can do this in parallel
- Suppose the answer mod m_j is a_j:
  - Use CRT to find x such that $x \equiv a_j \pmod{m_j}$
  - The unique x such that $0 < x < m_1 \cdots m_n$ is the answer to the original problem.

Example: The following are pairwise relatively prime:

$$2^{35} - 1,\ 2^{34} - 1,\ 2^{33} - 1,\ 2^{29} - 1,\ 2^{23} - 1$$

We can add and multiply positive integers up to

$$(2^{35} - 1)(2^{34} - 1)(2^{33} - 1)(2^{29} - 1)(2^{23} - 1) > 2^{163}.$$

Fermat's Little Theorem

Theorem 11 (Fermat's Little Theorem):
(a) If p prime and $\gcd(p, a) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$.

(b) For all $a \in Z$, $a^p \equiv a \pmod{p}$.

Proof. Let

$A = \{1, 2, \ldots, p-1\}$
$B = \{1a \bmod p, 2a \bmod p, \ldots, (p-1)a \bmod p\}$

Claim: $A = B$.

- $0 \notin B$, since $p \not| ja$, so $B \subset A$.
- If $i \neq j$, then $ia \bmod p \neq ja \bmod p$
  - since $p \not| (j - i)a$

Thus $|A| = p - 1$, so $A = B$.

Therefore,

$\Pi_{i \in A}\, i \equiv \Pi_{i \in B}\, i \pmod{p}$
$\Rightarrow (p-1)! \equiv a(2a) \cdots (p-1)a = (p-1)!\, a^{p-1} \pmod{p}$
$\Rightarrow p \mid (a^{p-1} - 1)(p-1)!$
$\Rightarrow p \mid (a^{p-1} - 1)$ [since $\gcd(p, (p-1)!) = 1$]
$\Rightarrow a^{p-1} \equiv 1 \pmod{p}$

It follows that $a^p \equiv a \pmod{p}$

- This is true even if $\gcd(p, a) \neq 1$; i.e., if $p \mid a$

Why is this being taught in a CS course?

Private Key Cryptography

Alice (aka A) wants to send an encrypted message to Bob (aka B).

- A and B might share a private key known only to them.
- The same key serves for encryption and decryption.
- Example: Caesar's cipher $f(m) = m + 3 \bmod 26$ (shift each letter by three)
  - `WKH EXWOHU GLG LW`
  - `THE BUTLER DID IT`

This particular cryptosystem is very easy to solve

- Idea: look for common letters (E, A, T, S)

One Time Pads

Some private key systems are completely immune to cryptanalysis:

- A and B share the only two copies of a long list of random integers s_i for $i = 1, \ldots, N$.
- A sends B the message $\{m_i\}_{i=1}^n$ encrypted as:

$$c_i = (m_i + s_i) \bmod 26$$

- B decrypts A's message by computing $c_i - s_i \bmod 26$.

The good news: bulletproof cryptography
The bad news: horrible for e-commerce

- How do random users exchange the pad?

Public Key Cryptography

Idea of *public key cryptography* (Diffie-Hellman)

- Everyone's encryption scheme is posted publically
  - e.g. in a "telephone book"
- If A wants to send an encoded message to B, she looks up B's public key (i.e., B's encryption algorithm) in the telephone book
- But only B has the decryption key corresponding to his public key

BIG advantage: A need not know nor trust B.

There seems to be a problem though:

- If we publish the encryption key, won't everyone be able to decrypt?

Key observation: decrypting might be too hard, unless you know the key

- Computing f^{-1} could be much harder than computing f

Now the problem is to find an appropriate (f, f^{-1}) pair for which this is true

- Number theory to the rescue

RSA: Key Generation

Generating encryption/decryption keys

- Choose two very large (hundreds of digits) primes p, q.
  - This is done using probabilistic primality testing
  - Choose a random large number and check if it is prime
  - By the prime number theorem, there are lots of primes out there
- Let $n = pq$.
- Choose $e \in N$ relatively prime to $(p-1)(q-1)$.
  - How do you find e?: Guess e, and use Euclid's algorithm to check $\gcd(e, (p-1)(q-1)) = 1$
  - How many numbers less than n are relatively prime to $(p-1)(q-1)$?
    - Lots: could choose e to be another prime.
- Compute d, the inverse of e modulo $(p-1)(q-1)$.
  - Can do this using using Euclidean algorithm
- Publish n and e (that's your public key)
- Keep the decryption key d to yourself.

RSA: Sending encrypted messages

How does someone send you a message?

- The message is divided into blocks each represented as a number M between 0 and n. To encrypt M, send

$$C = M^e \bmod n.$$

  - Need to use fast exponentiation ($2\log(n)$ multiplications) to do this efficiently

Example: Encrypt "stop" using $e = 13$ and $n = 2537$:

- `s t o p` $\leftrightarrow$ `18 19 14 15` $\leftrightarrow$ `1819 1415`
- $1819^{13} \bmod 2537 = 2081$ and
  $1415^{13} \bmod 2537 = 2182$ so
- `2081 2182` is the encrypted message.
- We did not need to know $p = 43, q = 59$ for that.

RSA: Decryption

If you get an encrypted message $C = M^e \bmod n$, how do you decrypt

- Compute $C^d \equiv M^{ed} \pmod{n}$.
  - Can do this quickly using fast exponentiation again

Claim: $M^{ed} \equiv M \pmod{n}$

Proof: Since $ed \equiv 1 \pmod{(p-1)(q-1)}$

- $ed \equiv 1 \pmod{p-1}$ and $ed \equiv 1 \pmod{q-1}$

Since $ed = k(p-1)+1$ for some k,

$$M^{ed} = (M^{p-1})^k M \equiv M \pmod{p}$$

(Fermat's Little Theorem)

- True even if $p \mid M$

Similarly, $M^{ed} \equiv M \pmod{q}$

Since p, q, relatively prime, $M^{ed} \equiv M \pmod{n}$ (Theorem 10).

Note: Decryption would be easy for someone who can factor N.

- RSA depends on factoring being hard!

Digital Signatures

How can I send you a message in such a way that you're convinced it came from me (and can convince others).

- Want an analogue of a "certified" signature

Cool observation:

- To send a message M, send $M^d \pmod{n}$
  - where (n, e) is my public key
- Recipient (and anyone else) can compute $(M^d)^e \equiv M \pmod{n}$, since M is public
- No one else could have sent this message, since no one else knows d.

Probabilistic Primality Testing

RSA requires really large primes.

- This requires testing numbers for primality.
  - Although there are now polynomial tests, the standard approach now uses probabilistic primality tests

Main idea in probabilistic primality testing algorithm:

- Choose b between 1 and n at random
- Apply an easily computable (deterministic) test $T(b, n)$ such that
  - $T(b, n)$ is true (for all b) if n is prime.
  - $T(b, n)$ there are lots of b's for which b is false if n is not prime.

Example: Compute $\gcd(b, n)$.

- If n is prime, $\gcd(b, n) = 1$
- If n is composite, $\gcd(b, n) \neq 1$ for some b's
  - Problem: there may not be that many witnesses

Example: Compute $b^{n-1} \bmod n$

- If n is prime $b^{n-1} \equiv 1 \pmod{n}$ (Fermat)
- Unfortunately, there are some composite numbers n such that $b^{n-1} \equiv 1 \pmod{n}$
  - These are called *Carmichael numbers*

There are tests $T(b, n)$ with the property that

- $T(b, n) = 1$ for all b if n is prime
- $T(b, n) = 0$ for at least $1/3$ of the b's if n is composite
- $T(b, n)$ is computable quickly (in polynomial time)

Constructing T requires a little more number theory

- Beyond the scope of this course.

Given such a test T, it's easy to construct a probabilistic primality test:

- Choose 100 (or 200) b's at random
- Test $T(b, n)$ for each one
- If $T(b, n) = 0$ for any b, declare b composite
  - This is definitely correct
- If $T(b, n) = 1$ for all b's you chose, declare n prime
  - This is highly likely to be correct

Prelim Coverage

- Chapter 0:
  - Sets
    - Operations: union, intersection, complementation, set difference
    - Proving equality of sets
  - Relations:
    - reflexive, symmetric, transitive, equivalence relations
    - transitive closure
  - Functions
    - Injective, surjective, bijective
    - Inverse function
  - Important functions and how to manipulate them:
    - exponent, logarithms, ceiling, floor, mod
  - Summation and product notation
  - Matrices (especially how to multiply them)
  - Proof and logic concepts
    - logical notions ($\Rightarrow$, $\equiv$, $\neg$)
    - Proofs by contradiction

- Chapter 1
  - You don't have to write algorithms in their notation
  - You may have to *read* algorithms in their notation
- Chapter 2
  - induction vs. strong induction
  - guessing the right inductive hypothesis
  - inductive (recursive) definitions
- Number Theory - everything we covered in class including
  - Fundamental Theorem of Arithmetic
  - gcd, lcm
  - Euclid's Algorithm and its extended version
  - Modular arithmetic, linear congruences,
  - modular inverse and CRT
  - Fermat's little theorem
  - RSA
  - Probabilistic primality testing

  You need to know all the theorems and corollaries discussed in class.