Semantics of First-Order Logic

Assume we have some domain D.

- The domain could be finite:
  - $\{1, 2, 3, 4, 5\}$
  - the people in this room
- The domain could be infinite
  - N, R, ...

A statement like $\forall x P(x)$, means that $P(d)$ is true for each d in the domain.

- If the domain is N, then $\forall x P(x)$ is equivalent to

$$P(1) \wedge P(2) \wedge \ldots$$

Similarly, $\exists x P(x)$ means that $P(d)$ is true for some d in the domain.

- If the domain is N, then $\exists x P(x)$ is equivalent to

$$P(1) \vee P(2) \vee \ldots$$

Is $\exists x(x^2 = 2)$ true?

Yes if the domain is R; no if the domain is N.

How about $\forall x \forall y((x < y) \Rightarrow \exists z(x < z < y))$?

First-Order Logic: Formal Semantics

How do we decide if a first-order formula is true? Need:

- a domain D (what are you quantifying over)
- an *interpretation* I that interprets the constants and predicate symbols:
  - for each constant symbol c, $I(c) \in D$
    - Which domain element is Alice?
  - for each unary predicate P, $I(P)$ is a predicate on domain D
    - formally, $I(P)(d) \in \{\text{true,false}\}$ for each $d \in D$
    - Is Alice Tall? How about Bob?
  - for each binary predicate Q, $I(Q)$ is a predicate on $D \times D$:
    - formally, $I(Q)(d_1, d_2) \in \{\text{true,false}\}$ for each $d_1, d_2 \in D$
    - Is Alice taller than Bob?
- a valuation V associating with each variable x an element $V(x) \in D$.
  - To figure out if $P(x)$ is true, you need to know what x is.

Now we can define whether a formula A is true, given a domain D, an interpretation I, and a valuation V, written

$$(I, D, V) \models A$$

- Read this from right to left, like Hebrew: A is true at $(\models)$ (I, D, V)

The definition is by induction:

$(I, D, V) \models P(x)$ if $I(P)(V(x)) = \text{true}$
$(I, D, V) \models P(c)$ if $I(P)(I(c))) = \text{true}$
$(I, D, V) \models \forall x A$ if $(I, D, V') \models A$ for all valuations V' that agree with V except possibly on x

- $V'(y) = V(y)$ for all $y \neq x$
- $V'(x)$ can be arbitrary

$(I, D, V) \models \exists x A$ if $(I, D, V') \models A$ for some valuation V' that agrees with V except possibly on x.

Translating from English to First-Order Logic

All men are mortal
Socrates is a man
Therefore Socrates is mortal

There is two unary predicates: *Mortal* and *Man*
There is one constant: *Socrates*
The domain is the set of all people

$$\forall x(Man(x) \Rightarrow Mortal(x))$$
$$Man(Socrates)$$

$$Mortal(Socrates)$$

More on Quantifiers

$\forall x \forall y P(x, y)$ is equivalent to $\forall y \forall x P(x, y)$

- P is true for every choice of x and y

Similarly $\exists x \exists y P(x, y)$ is equivalent to $\exists y \exists x P(x, y)$

- P is true for some choice of (x, y).

What about $\forall x \exists y P(x, y)$? Is it equivalent to $\exists y \forall x P(x, y)$?

- Suppose the domain is the natural numbers. Compare:
  - $\forall x \exists y (y \geq x)$
  - $\exists y \forall x (y \geq x)$

In general, $\exists y \forall x P(x, y) \Rightarrow \forall x \exists y P(x, y)$ is *logically valid.*

- A logically valid formula in first-order logic is the analogue of a tautology in propositional logic.
- A formula is logically valid if it's true in every domain and for every *interpretation* of the predicate symbols.

Bound and Free Variables

$\forall i(i^2 > i)$ is equivalent to $\forall j(j^2 > j)$:

- the i and j are *bound* variables, just like the i, j in

$$\sum_{i=1}^{n} i^2 \text{ or } \sum_{j=1}^{n} j^2$$

What about $\exists i(i^2 = j)$:

- the i is bound by $\exists i$; the j is *free*. Its value is unconstrained.
- if the domain is the natural numbers, the truth of this formula depends on the value of j.

Axiomatizing First-Order Logic

Just as in propositional logic, there are axioms and rules of inference that provide a sound and complete axiomatization for first-order logic, independent of the domain.

A typical axiom:

- $\forall x(P(x) \Rightarrow Q(x)) \Rightarrow (\forall x P(x) \Rightarrow \forall x Q(x))$.

A typical rule of inference is *Universal Generalization*:

$$\frac{\varphi(x)}{\forall x \varphi(x)}$$

Gödel proved completeness of this axiom system in 1930.

Axiomatizing Arithmetic

Suppose we restrict the domain to the natural numbers, and allow only the standard symbols of arithmetic ($+$, $\times$, $=$, $>$, 0, 1). Typical true formulas include:

- $\forall x \exists y (x \times y = x)$
- $\forall x \exists y (x = y + y \vee x = y + y + 1)$

Let $Prime(x)$ be an abbreviation for

$$\forall y \forall z ((x = y \times z) \Rightarrow ((y = 1) \vee (y = x)))$$

- $Prime(x)$ is true if x is prime

What does the following formula say:

- $\forall x (\exists y (y > 1 \wedge x = y + y) \Rightarrow$ $\exists z_1 \exists z_2 (Prime(z_1) \wedge Prime(z_2) \wedge x = z_1 + z_2))$
- This is *Goldbach's conjecture*: every even number other than 2 is the sum of two primes.
  - Is it true? We don't know.

Is there a nice (technically: recursive, so that a program can check whether a formula is an axiom) sound and complete axiomatization for arithmetic?

- *Gödel's Incompleteness Theorem:* NO!

Logic: The Big Picture

A typical logic is described in terms of

- *syntax*: what are the legitimate formulas
- *semantics*: under what circumstances is a formula true
- *proof theory/ axiomatization*: rules for proving a formula true

Truth and provability are quite different.

- What is provable depends on the axioms and inference rules you use
- Provability is a mechanical, turn-the-crank process
- What is true depends on the semantics

Tautologies and Valid Arguments

When is an argument

A_1
A_2
$\vdots$
A_n
———
B

valid?

- When the truth of the premises imply the truth of the conclusion

How do you check if an argument is valid?

- Method 1: Take an arbitrary truth assignment v. Show that if $A_1, \ldots, A_n$ are true under v ($v \models A_1$, $\ldots v \models A_n$) then B is true under v.
- Method 2: Show that $A_1 \wedge \ldots \wedge A_n \Rightarrow B$ is a tautology (essentially the same as Method 1)
  - true for every truth assignment
- Method 3: Try to prove $A_1 \wedge \ldots \wedge A_n \Rightarrow B$ using a sound axiomatization