CS 280 - Homework 7
Solutions

Problem 1. (6)

a. Basically everyone got this. The formula is just

$$\frac{\binom{5}{2}\binom{10}{2}\binom{15}{2}\binom{5}{2}}{\binom{35}{8}}$$

As the problem was incorrect in stating that there were 30 people (before, it said there were 32), answers with 30 and 32 in the denominator were accepted.

b. For both parts it does not make a difference what order you pick in. The probability for the first is $\frac{1}{18}$ for each, while the probability is $\frac{1}{9}$ for the second. If you go through the math carefully you should arrive at those probabilities.

One common mistakes was not realizing that the ball could already have been picked by the time you get to the end, meaning that the probability is not 1 for picking last! Also, for some reason a lot of people thought that picking 5th for the second part was best. Again, this is not the case.

c. From definition 1 on page 452 of the text book, using P as the probability that there were 3 purple flowers and B as the probability that the bucket picked was B, we get for part (i):

$$\begin{aligned} Pr(B \mid P) &= \frac{Pr(P \cap B)}{Pr(P)} \\ &= \frac{\frac{12}{24}\frac{11}{23}\frac{10}{22}\frac{1}{2}}{\frac{12}{24}\frac{11}{23}\frac{10}{22}\frac{1}{2} + \frac{8}{24}\frac{7}{23}\frac{6}{22}\frac{1}{2}} \\ &= \frac{55}{69} \\ &\approx .7971 \end{aligned}$$

Similarly for part (ii) we get, with D as the probability that all different:

$$\begin{aligned} Pr(D \mid P) &= \frac{Pr(D \cap B)}{Pr(D)} \\ &= \frac{\frac{12}{24}\frac{6}{23}\frac{6}{22}\frac{1}{2}}{\frac{12}{24}\frac{6}{23}\frac{6}{22}\frac{1}{2} + \frac{8}{24}\frac{8}{23}\frac{8}{22}\frac{1}{2}} \\ &= \frac{32}{59} \\ &\approx .5424 \end{aligned}$$

Problem 2. (4)

a. The probability that n people all have different birthdays is

$$f(n) = \prod_{k=1}^{n-1} \left(1 - \frac{k}{365}\right)$$

If you do this multiplication out, you see that the first n such that $f(n) < \frac{1}{2}$ is $n = 23$.

b. See `http://www.physics.harvard.edu/probweek/sol16.pdf`.

Problem 3. (4)

a. The binomial distribution function is

$$B(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

We need to find the value of k that maximizes this function.

Since the shape of the distribution curve has only one peak, there is only one point where the distribution stops increasing and starts decreasing. This means that at the maximal point, the ratio between successive terms changes from a value greater than 1 to a value less than 1.

Note that this process is comparable to finding the point at which a continuous function's derivative becomes zero.

The ratio between successive terms is:

$$\begin{aligned} \frac{B(k+1)}{B(k)} &= \frac{\binom{n}{k+1} p^{k+1}(1-p)^{n-(k+1)}}{\binom{n}{k} p^k (1-p)^{n-k}} \\ &= \frac{n! p^{k+1}(1-p)^{n-(k+1)}}{\frac{(k+1)!(n-(k+1))!}{n! p^k (1-p)^{n-k}} k!(n-k)!} \\ &= \frac{p(n-k)}{(1-p)(k+1)} \end{aligned}$$

We want to find the smallest value of k that satisfies the inequality:

$$\frac{B(k+1)}{B(k)} < 1$$

Then this value of k will be the mode (the point having maximum probability).

$$\begin{aligned} \frac{p(n-k)}{(1-p)(k+1)} &< 1 \\ \frac{pn - pk}{k+1-pk-p} &< 1 \\ pn - pk &< k + 1 - pk - p & \quad [k - pk > 0 \text{ and } 1 - p > 0 \text{ so } k + 1 - pk - p > 0, \\ k(-p-1+p) &< 1 - p - pn & \text{and we don't have to change the } < \text{ to } >] \\ k &> -1 + p + pn \end{aligned}$$

The smallest value of k that satisfies this inequality is $p(n+1) - 1$. Since k can only take on integer values, we must round up, so $k = \lceil p(n+1) - 1 \rceil = \lfloor p(n+1) \rfloor$

b. Nearly everybody got this problem right, so I'll keep this short.
In order for you to believe your friend, all 5 of 5 flips must come up the same.

i. If $p = \frac{1}{2}$, the probability of all five flips coming up heads is $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$. Likewise, the probability of all five flips coming up tails is $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$. Since these events are independent, the probability of either happening is just the sum $\frac{1}{32} + \frac{1}{32} = \frac{1}{16}$.

ii. If $p = \frac{3}{4}$, the probability of all five flips coming up heads is $\left(\frac{3}{4}\right)^5 = \frac{243}{1024}$. The probability of all five flips coming up tails is just $\left(1 - \frac{3}{4}\right)^5 = \frac{1}{1024}$. The probability of all five flips being the same is $\frac{243}{1024} + \frac{1}{1024} = \frac{244}{1024} = \frac{61}{256}$.

iii. The probability of acceptance, as a function of p, is: $\text{Prob}(p) = p^5 + (1-p)^5$.

Problem 4. (6)

a. The pdf of $x1$, $x2$ and $x3$ is as follows: Note that $x1$ and $x2$ determine $x3$, so really on two variables at a time can be in the equation, because determining their values automatically determines the last variable's value as well. The same applies to $p1$, $p2$ and $p3$:

$$f(x_1, x_2, x_3) = \frac{100!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{100 - x_1 - x_2}$$

b. In order to derive $f_1(x_1)$ we can express as the following equation:

$$f_1(x_1) = \sum_{x_2=0}^{100-x^1} \frac{100!}{x_1!x_2!(100 - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{100 - x_1 - x_2}$$

However, we want a marginal distribution, so the answer is

$$F(a) = \sum_{k=0}^{a} f_1(k)$$

c. Now we have $f_2(x_2|x_1)$ as the function we wish to calculate. This is equivalent to $\frac{f(x_1,x_2)}{f(x_1)}$ The numerator is

$$\frac{100!}{x_1! * x_2! * (100 - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{100 - x_1 - x_2}$$

The denominator is

$$\frac{100!}{x_1!(100 - x_1)!} p_1^{x_1} (p_2 + p_3)^{100 - x_1}$$

Dividing, we get

$$f_2(x_2|x_1) = \frac{(100 - x_1)!}{x_2(100 - x_1 - x_2)!} \frac{p_2^{x_2} p_3^{10 - x_2}}{(p_2 + p_3)10} = \binom{100 - x_1}{x2} \frac{p_2^{x_2} p_3^{10 - x_2}}{(p2 + p3)10}$$

Remember that we want a distribution, so we sum the above function to get

$$F(a) = \sum_{x_2=0}^{a} f_2(x_2|x_1 = 90)$$

Problem 5. (4) No, pairwise independence does not imply mutual independence. A counter-example: Let S be $\{-1, 1\}$ for x_1, x_2, and x_3. Define x_1 and x_2 to be independent, where the pdf for each of these variables is as follows:

$$f_1(-1) = \frac{1}{2} \qquad f_1(1) = \frac{1}{2}$$
$$f_2(-1) = \frac{1}{2} \qquad f_2(1) = \frac{1}{2}$$

Define $x_3 = x_1 \cdot x_2$, so that $x_3 = 1$ when $x_1 = 1, x_2 = 1$ or $x_1 = -1, x_2 = -1$. Similarly, $x_3 = -1$ when $x_1 = 1, x_2 = -1$ or $x_1 = -1, x_2 = 1$.

Note that we have defined x_1 and x_2 to be independent; that is, we know that $f_{12}(x_1, x_2) = f_1(x_1)f_2(x_2)$. Clearly x_1 and x_3 are independent — if x_1 is fixed, then x_3 varies with x_2; so, since x_1 and x_2 are independent, so are x_1 and x_3. By a similar argument, x_2 and x_3 are independent.

Despite this, however, x_1, x_2, and x_3 are not mutually independent. This is obvious, but it can be illustrated by an example. The probability $f(1, 1, -1) = 0$, since $x_3 = 1$ whenever $x_1 = x_2 = 1$. But $f_1(1)f_2(1)f_3(-1) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \neq 0$. Thus $f(x_1, x_2, x_3) \neq f_1(x_1)f_2(x_2)f_3(x_3)$, and pairwise independence does not imply mutual independence.

Problem 6. (6)

a.

$$\mu_k^* = E\left[\frac{x!}{(x-k)!}\right] = \sum_{x=0}^{\infty} \frac{x!}{(x-k)!} f(x) = \sum_{x=0}^{\infty} \frac{x!}{(x-k)!} \frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-k)!}$$

b. The mean is $E[x] = \mu_1^*$. The variance is $E[x^2] - (E[x])^2$. Note that

$$\mu_2^* = E[x(x-1)] = E[x^2 - x] = E[x^2] - E[x]$$

Thus

$$\sigma^2 = \mu_2^* + E[x] - E[x]^2 = \mu_2^* + \mu_1^* - (\mu_1^*)^2$$

c) Simply substitute the hypergeometric pdf in for $f(x)$ in

$$E\left[\frac{x!}{(x-k)!}\right] = \sum_{x=0}^{\infty} \frac{x!}{(x-k)!} f(x)$$

then calculate this for $k = 1$, $k = 2$ and substitue into the result from (b).