

CS 280 Solution Guide

Homework 5

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1(a). How many ways are there to rearrange the letters in the word COMPUTER?

There are 8 distinct letters in the word COMPUTER. Therefore, the number of ways to rearrange the letters would simply be $8! = 40\,320$

1(b). How many ways if you are required to keep the vowels in alphabetical order? (In other words, the consonants can be in any order at all, but the vowels must appear in the order EOU, possibly with consonants in between the vowels.)

In any rearrangement of the word COMPUTER, there are $3!$ ways to permute the vowels in the word. Only 1 of out these $3!$ ways are correct. Therefore, out of the $8!$ possible arrangement of the letters in the word COMPUTER, only $1/3!$ of these arrangements have the 3 vowels in alphabetical order. Thus, the number of ways to arrange COMPUTER, keeping the vowels in alphabetical order, is $8!/3!$ (or $P(8, 3)$) = 6720

Another way to think about the problem is to consider the number of ways to permute the *consonants* first. There are 5 consonants and 8 positions to choose from. The number of ways to permute this is $8 \times 7 \times 6 \times 5 \times 4 = 6720$. There is only 1 way to arrange the vowels in the remaining 3 positions (namely, in the order E, O and U). Therefore, the number of arrangements given the constraints is 6720.

1(c). How many ways if the consonants must be kept in alphabetical order?

This is similar to question 1(b), except that there are $5!$ ways to permute the consonants in the word. Therefore, the number of ways to arrange COMPUTER, keeping the consonants in in alphabetical order is $8!/5! = 336$. Alternatively, we can first consider the number of ways to permute the vowels first. There are 3 vowels and 8 positions to choose from; the number of ways to permute this is $8 \times 7 \times 6 = 336$. There is only 1 way to arrange the consonants in the remaining 5 positions, so that number of arrangements given the constraint is still 336.

1(d). How many ways if both vowels and consonants must be kept in alphabetical order?

There are $C(8,5) = 56$ ways to choose 5 positions out of 8 on which the consonants must stay. Once these 5 positions are chosen, there is only 1 way to arrange the consonants in these 5 positions (i.e. in alphabetical order). Similarly, there is only 1 way to arrange the vowels in the remaining 3 positions. Therefore, there are 56 ways to arrange COMPUTER, given the above constraint.

Alternatively we can choose to position the vowels first. We will get $C(8,3) = 56$ ways to select a combination of positions on which the vowels must stay. After that, there is only 1 way to arrange all the vowels and consonants, resulting in the same answer.

2. Given an arbitrary set of 10 integers, prove that there is a subset of these 10 integers whose sum is divisible by 10.

Suppose the 10 integers are x_1, x_2, \dots, x_{10} .

Consider the 10 possible subsets $\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \dots, \{x_1, x_2, \dots, x_{10}\}$. We shall call them S_1, S_2, \dots, S_{10} in that order.

Consider the *last digit* of the sum of integers in each of the 10 subsets. If the last digit of the sum is 0, it indicates that the sum of integers in the subset is divisible by 10.

If any of the last digit of the sum of integers in each of the 10 subsets given above is 0, then, clearly, the sum is divisible by 10, and we are done.

Otherwise, the last digit of the sum of integers in each of the 10 subsets given above must be non-zero. There are only 9 possible non-zero digits, and 10 sums. By Pigeon Hole Principle, there must be 2 distinct subsets S_i and S_j for which their sums have the same last digits. Suppose that $j > i$ (this is arbitrary). Since the subsets are *monotone-increasing*, (i.e. $S_i \subseteq S_j$ if $i < j$), the sum of integers in the subset $S_j - S_i$ will be equal to (the sum of integers in subset S_j) - (the sum of integers in subset S_i). Since these 2 sums have the same last digits, the sum of integers in the subset $S_j - S_i$ will have a last digit of 0. Therefore, the subset $S_j - S_i$ will have a sum divisible by 10. Since $S_j - S_i$ is a valid subset of these 10 integers and it has a sum divisible by 10, we have shown that even if none of the subsets S_1, \dots, S_{10} are divisible by 10, there exists another subset ($S_j - S_i$) which is.

Therefore, for any arbitrary set of 10 integers, there is always a subset of these 10 integers whose sum is divisible by 10.

3. Five points with integer coordinates are plotted on a graph. Consider the midpoints of the segments that join these 5 points. Prove that at least one of these midpoints has integer coordinates.

Consider 2 points (x_i, y_i) and (x_j, y_j) .

The midpoint of these 2 points is $\frac{1}{2}(x_i + x_j, y_i + y_j)$

The x coordinate of this midpoint is an integer if and only if:

x_i and x_j are both odd or both even (same parity).

Similarly, the y coordinate of this midpoint is an integer if and only if:

y_i and y_j are both odd or both even (same parity).

Therefore, the midpoint of any 2 end-points has integer coordinates if the x -coordinates of both end-points have the same parity, and the y -coordinates of both end-points also have the same parity.

However, there are only $2^2 = 4$ possible ways to permute the parities of the x and y -coordinates of a point; namely (O, O), (O, E), (E, O) and (E, E). (O = odd, E = even)

Since there are 5 points and only 4 possible parity permutations, by the Pigeon Hole Principle, at least 2 points will have exactly the same parity for both the x -coordinates

and the y -coordinates. Subsequently, as proven above, the midpoint of these 2 points must have integer coordinates. Therefore, at least one of the midpoints has integer coordinates.

4(a). How many ways are there to divide 15 people into 3 teams of 5 each?

There are $C(15,5)$ ways of choosing a first team of 5 from 15 people.

There are $C(10,5)$ ways of choosing a second team of 5 from the 10 remaining people.

There are $C(5,5)$ ways of choosing a third team of 5 from the 5 remaining people.

Finally, there are $3!$ ways of permutating the order in which these 3 teams are chosen. In the above example, we have defined 3 separate teams chosen in a particular order, but if the people in these 3 teams are chosen in some other arbitrary order (e.g. people in team 2 gets chosen before the people in team 1), the permutation would essentially be the same, although we would have overcounted this case as another new way of dividing the 15 people.

Therefore, the number of ways to divide 15 people into 3 teams of 5 each is:

$$C(15,5) \times C(10,5) \times C(5,5) / 3! = (3003 \times 252 \times 1) / 6 = 126\,126.$$

4(b). How many ways are there to choose two teams of 5 people each from 15 people?

Similar to the above question:

There are $C(15,5)$ ways of choosing a first team of 5 from 15 people.

There are $C(10,5)$ ways of choosing a second team of 5 from the 10 remaining people.

There are $2! = 2$ ways of permutating the order in which the 2 teams are chosen.

Therefore, the number of ways to divide 15 people into 2 teams of 5 each is:

$$C(15,5) \times C(10,5) / 2! = (3003 \times 252) / 2 = 378\,378.$$

5(a). You're driving back to Ithaca and approaching the tollbooth as you exit the NY Thruway when you notice that there are two lines waiting to pay the toll: one of the line consisting entirely of m cars; the other consisting entirely of n trucks. Inspired by your love of cs280, you realize that you can find a formula for all the patterns by which the cars and trucks can leave the toll plaza. What is this formula? A pattern looks like $C_1T_1C_2C_3C_4T_2\dots$ where C_i represents the i^{th} car and T_i represents the i^{th} truck. You should assume that the cars remain in their current order and the trucks remain in their current order. The only thing that can vary is when the attendant decides to release the vehicle.

There are $m + n$ vehicles going through the tollbooth altogether.

Consider the pattern P generated by the sequence of vehicles leaving the tollbooth; it should have a length of $m + n$. Furthermore, of all the $m + n$ "bits" in the pattern P , m of them must be occupied by "C"s. The number of ways to choose m bits out of $m + n$ bits in P (for which the bit is occupied by a "C") is $C(m + n, m)$. Once chosen, there is only 1 way to arrange the m cars within the m chosen bits, and n trucks within n remaining bits; namely, in their order of arrival at the tollbooth.

Alternatively, we can choose to permute the trucks first, and get the answer $C(m + n, n)$. It has the same value as $C(m + n, m)$. (See the answer to question 1(d) for similar analogy)

5(b). After leaving the tollbooth, you continue on your journey to Ithaca, arriving late at night and hungry. You decide to buy a snack at P&C. At P&C, there are three lines open. There are people waiting in each line; in fact, there are p people in the first line, q people in the second and r people in the third. Find a formula for the number of possible orders in which these $p+q+r$ people leave P&C and give a brief explanation of why your formula is correct. You should assume that if any line becomes empty then the checker immediately closes that line; thus, people do not change lines even if another line becomes empty. Each person leaves the store as soon as they're checked out.

There are $p + q + r$ people leaving P&C altogether.

Let A_1, A_2, \dots, A_p denote the people waiting in the first line.

Let B_1, B_2, \dots, B_q denote the people waiting in the second line.

Let C_1, C_2, \dots, C_r denote the people waiting in the third line.

Consider the pattern P_2 generated by the sequence of people leaving the tollbooth; it should have a length of $p + q + r$. Furthermore, of all the $p + q + r$ "bits" in the pattern P_2 , p of them must be occupied by "A"s, denoting the people in the first line. The number of ways to choose p bits out of $p + q + r$ bits in P_2 (for which the bit is occupied by a "A") is $C(p + q + r, p)$. Furthermore, there are $C(q + r, q)$ ways to choose q bits out of the remaining $q + r$ bits (for which the bit is occupied by a "B"), and $C(r, r) = 1$ way to choose r bits from the remaining r bits available. Therefore, there are:

$C(p + q + r, p) \times C(q + r, q) \times C(r, r) = (p + q + r)! / (p! q! r!)$ ways to arrange the bits in the pattern P_2 such that p of them are "A"s, q of them are "B"s and r of them are "C"s.

Once an arrangement is chosen, there is only 1 way to arrange the p people in the first queue within the p chosen bits, the q people in the second queue within the q chosen bits, and the r people in the third queue within the r chosen bits; namely, in their order of arrival at the line.

Since the number of possible permutations of P_2 corresponds to the number of possible orders in which the $p + q + r$ people leave P&C, and there are $(p + q + r)! / (p! q! r!)$ possible permutations of P_2 , there are $(p + q + r)! / (p! q! r!)$ possible orders.

6. The instructor for a small class (7 students) tells the students after an exam that if any 4 of them get together they'll have at least one correct solution to every problem. He goes on to claim that if any 3 of them get together there will be at least one problem for which each of the 3 has an incorrect answer. The exam has 30 questions. Show that the instructor's claim must be incorrect.

Consider the number of *distinct* groups of 3 that the students in the class can form.

There are $C(7, 3) = 35$ possible groups of 3 which can be formed.

If the instructor's claim is correct, each of these 35 groups will have at least one problem for which the entire group has an incorrect answer. Since there are only 30 exam

questions and 35 possible groups of 3, by Pigeon-Hole Principle, there will be at least 2 possible groups of 3, (call them G_1 and G_2) which have incorrect answers to the same exam question.

Consider the group $G = G_1 \cup G_2$; since G_1 and G_2 are distinct groups, they differ by at least one person, so $G_1 \cup G_2$ will have at least 4 people. However, since both groups G_1 and G_2 have incorrect answers to the same exam question, the new group G will also have incorrect answers to the same exam question.

Since $|G| \geq 4$, and G does not have all the correct answers to the exam, any subset of G consisting of 4 people will not have all the correct answers to the exam, contradicting the instructor's claim that if any 4 of them get together they'll have at least one correct solution to every problem.
