Part A

1. **Method 1:** By induction.
   Let \( P(n) \) be the proposition that \( 4^{n+1} + 5^{2n-1} \) is divisible by 21.

   **Base case:**
   For \( n = 1 \), \( 4^{n+1} + 5^{2n-1} = 16 + 5 = 21 \)
   \( \therefore P(1) \) is true.

   **Inductive Hypothesis:**
   Assume that \( P(n) \) is true, that is, \( 4^{n+1} + 5^{2n-1} \) is divisible by 21.
   \( \because 4^{n+1} + 5^{2n-1} = 21k \) for some positive integer \( k \).
   It must be shown that \( P(n+1) \), which states that \( 4^{n+2} + 5^{2n+1} \) is divisible by 21 must also be true under this assumption. This can be done since
   \[
   4^{n+2} + 5^{2n+1} = 4 \times 4^{n+1} + 25 \times 5^{2n-1}
   = 4 \times 4^{n+1} + 4 \times 5^{2n-1} + 21 \times 5^{2n-1}
   = 4 \times (4^{n+1} + 5^{2n-1}) + 21 \times 5^{2n-1}
   = 4 \times 21k + 21 \times 5^{2n-1} \quad \text{(by the inductive hypothesis)}
   = 21(4k + 5^{2n-1})
   \]
   This shows that if \( P(n) \) is true, then \( P(n+1) \) must also be true. This completes the inductive step of the proof and shows that \( 4^{n+1} + 5^{2n-1} \) is divisible by 21 for all positive integers \( n \).

   **Method 2:**
   Let \( S = 4^{n+1} + 5^{2n-1} \). To test if \( S \) is divisible by 21, we must test if \( S \) is divisible by both 3 and 7.

   Taking congruences modulo 3,
   \[
   S \equiv 4^{n+1} + 5^{2n-1} \quad \text{(mod 3)}
   \equiv 1^{n+1} + (-1)^{2n-1} \quad \text{(mod 3)}
   \equiv 1 + (-1) \quad \text{(mod 3)}
   \equiv 0 \quad \text{(mod 3)}
   \]
   \( \therefore S \) is divisible by 3.

   Taking congruences modulo 7,
   \[
   S \equiv 4^{n+1} + 5^{2n-1} \quad \text{(mod 7)}
   \equiv 16(4^{n-1}) + 5(25^{n-1}) \quad \text{(mod 7)}
   \equiv 2(4^{n-1}) + 5(4^{n-1}) \quad \text{(mod 7)}
   \equiv 7(4^{n-1}) \quad \text{(mod 7)}
   \equiv 0 \quad \text{(mod 7)}
   \]
   \( \therefore S \) is divisible by 7.
a. Obviously, 1000 does not have a single digit greater than 4, so we can consider the numbers from 1 to 1000 according to these 3 cases: 1 to 9, 10 to 99, and 100 to 999. As observed, the numbers within each of the 3 cases have exactly the number of digits.

∴ From 1 to 9, there are 5 numbers greater than 4. 
From 10 to 99, each of the 2 digits must be greater than 4, so there is a total of 5×5 (= 25) numbers with digits all greater than 4.
From 100 to 999, each of the 3 digits must be greater than 4, so there is a total of 5×5×5 (= 125) numbers with digits all greater than 4.

Adding up all the numbers from each of the 3 cases, we get 155 numbers between 1 and 1000 that have digits all greater than 4.

b. We will consider the same 3 cases for the same reasons.

∴ From 1 to 9, there are 4 numbers lesser than 5.
From 10 to 99, either only the first digit is less than 5 or the second digit is less than 5, but not both. If the first digit is less than 5, there are only 4 choices: 1, 2, 3, or 4. If the second digit is less than 5, there are 5 choices: 0, 1, 2, 3, or 4. This gives us (4×5 + 5×5 = 45) numbers with exactly one digit that is less than 5. Similarly, from 100 to 999, we must consider whether each digit is less than 5 separately. This gives us (4×5×5 + 5×5×5 + 5×5×5 = 350) numbers with exactly one digit that is less than 5.

Adding up all the numbers from each of the 3 cases, we get 399 numbers in total.

Part B
3 a. Consider the following list of number made up entirely of ones: 7, 77, 777, 7777,..., "7 repeated 1124 times" and consider their remainders (mod 1123). The list consists of 1124 items, but there are just 1123 possible remainders. Thus, by the Pigeonhole Principle, there must be two items in the list that have the same remainder (mod 1123). Their difference will be congruent to 0 (mod 1123) and thus divisible by 1123.

b. Note that the difference from the previous part looks like several sevens followed by several zeros. Now consider what happens when we divide by 10. This removes one zero from the end of the number, but does not change the fact that 1123 divides the number. This is because 1123 contains no factors of 2 or 5. We can continue removing zeros in this way to get a final number, consisting entirely of ones, that is divisible by 1123.

4 a. There are 50 possible scores. By setting the 126 students as 126 pigeons and the 50 scores as 50 pigeonholes, we can use the Pigeonhole Principle to conclude that there must be a pigeonhole containing at least 3 pigeons. Thus, there are at least 3 students who receive the same score.
b. There are 26 possible scores. Similarly, we can set the students as pigeons and
the scores as pigeonholes and use the Pigeonhole Principle to conclude that there
must be a pigeonhole containing at least 5 pigeons. Thus, we can conclude that 4
students share the same score.

By exactly the same reasoning, we can also conclude that 5 students share the
same score.

Part C

5 a. Let the 10 integers be \( x_1, x_2, \ldots x_{10} \). With these integers, consider the following
list of 10 sums: \( x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + \ldots + x_{10} \).

If any of these sums is divisible by 10, then we are done because we can choose
the integers that compose that sum to form the subset desired.

However, if none of the sums is divisible by 10, then we can consider their
remainders (mod 10). As stated, because none of the sums is divisible by 10,
therefore there are only 9 possible remainders. Because there are 10 total sums, by
the Pigeonhole Principle, there must be two sums in the list that have the same
remainder (mod 10). Their difference will be congruent to 0 (mod 10) and thus
divisible by 10. We can choose the integers that compose that difference to form
the subset desired.

b. Let \( S = \{1, 11, 21, 31, 41, 51, 61, 71, 81\} \). We note that each of the number in
S is congruent to 1 modulo 10, so for any nonempty subset of S the sum will be
congruent to \( n \) modulo 10, where \( n \) is the size of the subset. As there are only 9
numbers in S, this then necessarily means that \( 1 \leq n \leq 9 \). Therefore by construction,
we have shown that it is possible for there to be no subset whose sum is divisible
by 10 if there are just 9 integers.

6a. Let \( P(n) \) be the proposition that \( \sum_{i=1}^{n} (4i+1) = 2n^2 + 3n \)

Base case:
For \( n = 1 \), LHS = 4(1) + 1 = 5 = 2(1) + 3(1) = RHS
Therefore, \( P(1) \) is true.

Inductive Hypothesis:
Assume that \( P(n) \) is true, that is, \( \sum_{i=1}^{n} (4i+1) = 2n^2 + 3n \). To show that this implies
that \( P(n+1) \) is true, add \([4(n+1) + 1]\) to both sides of the equation to obtain:

\[
\sum_{i=1}^{n} (4i+1) + (4(n+1) + 1) = 2n^2 + 3n + (4(n+1) + 1) \quad \text{(by the inductive hypothesis)}
\]

\[
\therefore \sum_{i=1}^{n} (4i+1) = 2n^2 + 7n + 5
\]
\[= 2(n^2 + 2n + 1) + 3(n+1)\]
\[= 2(n + 1)^2 + 3(n+1)\]

This shows that if \(P(n)\) is true, then \(P(n+1)\) must also be true. This completes the inductive step of the proof and shows that the formula for the summation is true for all positive integers \(n\).

b. We can first reword the problem. Instead of showing that \(b_n\) is never divisible by 3, we can show that \(b_n \equiv 1 \pmod{3}\).

Let \(P(n)\) be the proposition that \(b_n \equiv 1 \pmod{3}\).

**Base cases:**
For \(n = 1\), \(b_1 = 1 \Rightarrow b_1 \equiv 1 \pmod{3}\)
∴ \(P(1)\) is true.

For \(n = 2\), \(b_2 = 1 \Rightarrow b_2 \equiv 1 \pmod{3}\)
∴ \(P(2)\) is true.

**Inductive Hypothesis:**
Assume that \(P(n)\) and \(P(n+1)\) are true, that is, \(b_n \equiv 1 \pmod{3}\) and \(b_{n+1} \equiv 1 \pmod{3}\).
It must be shown that \(P(n+2)\), which states that \(b_{n+2} \equiv 1 \pmod{3}\) must also be true under this assumption. This can be done since

\[b_{n+2} = 2b_n + 2b_{n+1}\]
\[\Rightarrow b_{n+2} \equiv 2b_n + 2b_{n+1} \pmod{3}\]
\[\equiv 2(1) + 2(1) \pmod{3}\] (by the inductive hypothesis)
\[\equiv 4 \pmod{3}\]
\[\equiv 1 \pmod{3}\]

This shows that if \(P(n)\) and \(P(n+1)\) are true, then \(P(n+2)\) must also be true. This completes the inductive step of the proof and shows that \(b_n \equiv 1 \pmod{3}\) for all positive integers \(n\).