1. (5 points; 1 each)
Let \( L(x, y) \) represent "\( x \) likes \( y \)". Let \( P(x) \) represent "\( x \) is a person". Let \( F(x) \) represent "\( x \) is a food". The universe is the set of all people, animals, and foods. Use quantifiers to express each of the following statements. Note that terms such as "everybody" and "nobody" refer to people.

a. Everybody who likes eggplant also likes chocolate.
\[
\forall x \left[ (P(x) \land L(x, \text{eggplant})) \rightarrow L(x, \text{chocolate}) \right]
\]
b. Nobody likes broccoli.
\[
\neg \exists x \left[ (P(x) \land L(x, \text{broccoli})) \right]
\]
c. There is a food that Elmo dislikes.
\[
\exists y \neg L(\text{Elmo}, y)
\]
d. Nobody likes every food.
\[
\neg \exists x \left[ (P(x) \land \forall y (F(y) \rightarrow L(x, y))) \right]
\]
e. Bert doesn't like anybody.
\[
\forall x [P(x) \rightarrow \neg L(\text{Bert}, x)]
\]

2. (3 points; 1 each)
Suppose \( f(x) = 5x - 1 \).

a. If \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) then is \( f \) onto? Explain (briefly).
   No. There is no \( x \) that hits 0.

b. If \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) then is \( f \) 1-to-1? Explain (briefly).
   Yes. \( f(x) = f(y) \) implies that \( 5x-1 = 5y-1 \). This can be simplified to show \( x=y \).

c. If \( f : \mathbb{R} \rightarrow \mathbb{R} \) then is \( f \) onto? Explain (briefly).
   Yes. For any real number \( a \), we can solve \( 5x-1 = a \). Thus, we hit all of \( \mathbb{R} \).

3. (4 points; 1 and 3)
Define \( a_n \)
\[
a_n = \begin{cases} 
1, & \text{if } n = 0 \text{ or } n = 1 \\
2a_{n-1} + 3a_{n-2}, & \text{otherwise.}
\end{cases}
\]

a. What is \( a_4 \)?
   \[a_0 = 1, a_1 = 1, a_2 = 5, a_3 = 13, a_4 = 41\]

b. Use induction to show that \( a_n \) is odd for all \( n \geq 0 \).
   Goal: Show \( P(n) \) holds for all \( n \geq 0 \) where \( P(n) \) is “\( a_n \) is odd”.
   Basis: \( P(0) \) and \( P(1) \) hold trivially since 1 is odd.
   Induction Hypothesis: (strong induction) \( P(k) \) holds for \( k < n \).
   Consider \( n > 1 \). By definition, \( a_n = 2a_{n-1} + 3a_{n-2} \).
   Using the Induction Hypothesis, we have \( a_n = 2\text{odd} + 3\text{odd}, \) which must be odd.
   Thus \( P(n) \) holds.
   Proof is complete.
4. (4 points)
Use induction to show that  \( \sum_{i=1}^{n} (2i-5) = n(n-4) \) for \( n>0 \).

**Goal:** Show \( P(n) \) holds for all \( n > 0 \) where \( P(n) \) is “\( \sum_{i=1}^{n} (2i-5) = n(n-4) \)”.

**Basis:** \( P(1) \) holds since \( 2*1 - 5 = (1)(1-4) = -3 \).

**Induction Hypothesis:** \( P(n) \) holds for some \( n \).

Consider \( n+1 \).

\[
\text{the sum to } n+1 = \text{(the sum to } n) + (2n+1 - 5) = n(n-4) + (2n+1 - 5) \text{ by the I.H.}
\]
\[
= n^2 - 4n + 2n - 3 = n^2 - 2n - 3 = (n + 1)(n - 3) = (n+1)((n+1) - 4).
\]

Thus \( P(n+1) \) holds.

**Proof is complete.**

5. (10 points; 1 each)
For each of the following statements about sets, mark the statement as either true or false. No proof is necessary. To be true the statement must hold for all sets \( A \) and \( B \); otherwise the statement is false.
\( P(A) \) represents the power set of \( A \). \( A^c \) represents the complement of \( A \).

a. \( A \cup (B \cap A^c) = A \cup B \)  \( \text{True} \)
b. \( \emptyset \in P(\emptyset) \)  \( \text{True} \)
c. \( A - \{\emptyset\} = A \)  \( \text{False} \)
d. \( (A \times B^c) \cup (A^c \times B) = [(A \times B) \cup (A^c \times B^c)]^c \)  \( \text{True} \)
e. \( \emptyset \subseteq P(\emptyset) \)  \( \text{True} \)

Which of the following are true for any positive functions \( f \) and \( g \) on the positive integers. No proof is necessary.
f. \( f(n) = O(g(n)) \) implies \( g(n) = \Omega(f(n)) \)  \( \text{True} \)
g. \( f(n) + g(n) = O(\min(f(n), g(n))) \)  \( \text{False} \)
h. \( \max(f(n), g(n)) = O(f(n) + g(n)) \)  \( \text{True} \)
i. \( f(n) = O(g(n)) \) implies 10n \( f(n) = O(n g(n)) \)  \( \text{True} \)
j. \( f(n) = O(n^2) \) and \( g(n) = O(n) \) implies \( f(n)/g(n) = O(n) \)  \( \text{False (Consider f(n) = n^2 and g(n) = 1.)} \)

6. (9 points; 3 each)
Solve for \( x \) in each of the following. For partial credit you must show your work.
a. 50 \( x \equiv 3 \) (mod 91)
\[
gcd(50, 91) = \gcd(50, 41) = \gcd(9, 41) = \gcd(9, 5) = \gcd(4, 5) = \gcd(4, 1) = 1
\]
\[
I = (0)(4) + (1)(1) = (0)(4) + (1)(5) = (-1)(9 - 5) + (1)(5)
\]
\[
= (-1)(9) + (2)(5) = (-1)(9) + (2)(41 - 4*9) = (-9)(9) + (2)(41) = (-9)(50 - 41) + (2)(41)
\]
\[
\]
Thus, we have \( I \equiv -20(50) \) (mod 91).

Multiplying both sides of the original congruence by \(-20 \), we get
\( x \equiv -60 \) (mod 91). Alternately, this can be written as \( x \equiv 31 \) (mod 91).

b. 8 \( x \equiv 2 \) (mod 17)
\[
gcd(8, 17) = \gcd(8, 1) = 1
\]
\[
I = (0)(8) + (1)(1) = (0)(8) + (1)(17 - 2*8) = (-2)(8) + (1)(17).
\]
Thus, we have \( I \equiv -2(8) \) (mod 17).

Multiplying both sides of the original congruence by \(-2 \), we get
\( x \equiv -4 \equiv 13 \) (mod 17).
c. \[ x \equiv 4 \pmod{9} \]
\[ x \equiv 3 \pmod{8} \]
\[ x \equiv 1 \pmod{5} \]

Find the smallest positive \( x \) for which all three congruences hold.

\[ b_1 = 40y \equiv 1 \pmod{9} \]. In other words, \( 4y \equiv 1 \pmod{9} \). A solution is \( y = 7 \); thus, \( b_1 = 280 \).

\[ b_2 = 45y \equiv 1 \pmod{8} \]. In other words, \( 5y \equiv 1 \pmod{8} \). A solution is \( y = 5 \); thus, \( b_2 = 225 \).

\[ b_3 = 72y \equiv 1 \pmod{5} \]. In other words, \( 2y \equiv 1 \pmod{5} \). A solution is \( y = 3 \); thus \( b_3 = 216 \).

A solution for \( x \) is thus \( 4*280 + 3*225 + 216 = 2011 \). To find the smallest solution, we take this (mod 360) to get 211.

7. (15 points; 3 each)
For partial credit, you must show your work.

a. Suppose \( a, b, \) and \( c \) are natural numbers, each greater than 1. Using a constructive proof, show that there is a number \( d \) that is not divisible by \( a \), not divisible by \( b \), and not divisible by \( c \).

Let \( d = abc + 1 \). When divided by \( a \), by \( b \), or by \( c \), the remainder is clearly 1, so it's not divisible by \( a \), by \( b \), or by \( c \).

b. What is \( 705 + (222 \cdot 123) + 8 + (1234567)^{1234567} \pmod{5} \)?

This is equivalent (mod 5) to \( 0 + (2*3) + 3 + 2^{1234567} \).

By testing powers of 2, we see that \( 2^4 \equiv 1 \pmod{5} \), so \( 2^{1234567} \equiv 2^{1234567 \pmod{4} = 2^3} \pmod{5} \).

Thus, the original expression is congruent to \( 9+8 = 17 \equiv 2 \pmod{5} \).

c. Prove or disprove: If \( \gcd(a, b) = \gcd(b, c) = r \) then \( \gcd(a, c) = r \).

False. Consider \( a = 18, b = 2, \) and \( c = 6 \). \( \gcd(a, b) = \gcd(b, c) = 2 \), but \( \gcd(a, c) = 6 \).

d. Prove or disprove: For all odd \( n \), \( 35^n + 637^n \equiv 0 \pmod{6} \).

\[ 35 \equiv -1 \pmod{6}. \ 637 \equiv 1 \pmod{6}. \ Thus, the expression is congruent to (-1)^n + (1)^n \pmod{6}. \]

It's easy to see that when \( n \) is odd this is 0.

e. Find a value \( n > 1 \) for which \( 2^n + 3^n + 5^n + 7^n + 2 \) is divisible by \( n \). Show that your choice for \( n \) works. If you think you need a calculator then you're probably using the wrong method.

If \( n \) were prime then the expression would be easy because we could use Fermat's Little Theorem \([a^{p-1} = a \pmod{p} \text{ for } p \text{ prime}]\). Assume for the moment that \( n \) is prime. Then the expression is congruent (mod \( n \)) to \( 2+3+5+7+2 = 19 \). With this reasoning to guide us, we test \( n = 19 \) and discover that it satisfies the requirements.