1.A Handout 11

11A. Section 2.6, 2b
\[ A + B = \begin{bmatrix} -1 & 0 & 5 & 6 \\ -4 & -3 & 5 & -2 \end{bmatrix} + \begin{bmatrix} -3 & 9 & -3 & 4 \\ 0 & -2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 9 & 2 & 10 \\ -4 & -5 & 4 & 0 \end{bmatrix} \]

11B. Section 2.6, 4c
\[ AB = \begin{bmatrix} 0 & -1 \\ 7 & 2 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} 4 & -1 & 2 & 3 & 0 \\ -2 & 0 & 3 & 4 & 1 \end{bmatrix} \]
\[ = \begin{bmatrix} (0 \cdot 4 + -1 \cdot -2) & (0 \cdot -1 + -1 \cdot 0) & (0 \cdot 2 + -1 \cdot 3) & (0 \cdot 3 + -1 \cdot 4) & (0 \cdot 0 + -1 \cdot 1) \\ (7 \cdot 4 + 2 \cdot -2) & (7 \cdot -1 + 2 \cdot 0) & (7 \cdot 2 + 2 \cdot 3) & (7 \cdot 3 + 2 \cdot 4) & (7 \cdot 0 + 2 \cdot 1) \\ (-4 \cdot 4 + -3 \cdot -2) & (-4 \cdot -1 + -3 \cdot 0) & (-4 \cdot 2 + -3 \cdot 3) & (-4 \cdot 3 + -3 \cdot 4) & (-4 \cdot 0 + -3 \cdot 1) \end{bmatrix} \]
\[ = \begin{bmatrix} 2 & 0 & -3 & -4 & -1 \\ 24 & -7 & 20 & 29 & 2 \\ -10 & 4 & -17 & -24 & -3 \end{bmatrix} \]

11C. Section 2.6, 24a
\[ A_1 \text{ is } 20 \times 50, \ A_2 \text{ is } 50 \times 10, \ A_3 \text{ is } 10 \times 40. \]
We examine the two possible cases. We will count only multiplications as they are more significant operations than addition (and this is the way the book makes these quantitative comparisons).

\((A_1A_2)A_3: \) Using the standard algorithm, \(20 \cdot 50 \cdot 10 = 10000\) multiplications are done for computing \((A_1A_2).\) Since this resulting matrix is \(20 \times 10\), the multiplication of it with \(A_3\) uses \(20 \times 10 \times 40 = 8000\) multiplications. Hence 18000 multiplications in all.

\(A_1(A_2A_3): \) \(50 \cdot 10 \cdot 40 = 20000\) multiplications are done for computing \((A_2A_3).\) Thus computing the product in this case is more expensive than the first case.

11D. Section 2.6, 28

(a) \[ A \lor B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \]

(b) \[ A \land B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \]

(c) \[ A \circ B = \begin{bmatrix} (1 \land 0) \lor (1 \land 1) & (1 \land 1) \lor (1 \land 0) \\ (0 \land 0) \lor (1 \land 1) & (0 \land 1) \lor (1 \land 0) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}. \]

1.B Handout 12

12A. Section 3.1, 2ade

(a) Simplification is used here.

(d) Addition is used here.

(e) Hypothetical syllogism used.

12B. Section 3.1, 10acd
(a) Let $P(x) = \text{“x owns a red convertible”}$, and $Q(x) = \text{“x has gotten a speeding ticket”}$.

Then we are asserting $P(\text{Linda})$, $\forall x P(x) \rightarrow Q(x)$, where the domain of $x$ is the set of students in the class. From these we may assert $P(\text{Linda}) \rightarrow Q(\text{Linda})$ by universal instantiation, then $Q(\text{Linda})$ by modus ponens, then $\exists x Q(x)$ by existential generalization.

(c) Let $P(x) = \text{“x is produced by John Sayles”}$, $Q(x) = \text{“x is wonderful”}$, and $R(x) = \text{“x is about coal miners”}$.

Then our assertions are that $\forall x P(x) \rightarrow Q(x)$, and $\exists x P(x) \land R(x)$, where the domain of $x$ is the set of movies. Then by existential instantiation, $P(c) \land R(c)$ for some movie $c$. By simplification, $P(c)$. By universal instantiation, $P(c) \rightarrow Q(c)$. Then our assertions are that $P(c)$, and $Q(c)$, by conjunction $P(c) \land R(c) \land Q(c)$. Finally by existential generalization, $\exists x P(x) \land R(x) \land Q(x)$.

(d) Let $P(x) = \text{“x has been to France”}$, $Q(x) = \text{“x has visited the Louvre”}$.

Then we assert the propositions $p_1 = \exists x P(x)$, and $q = \forall y P(y) \rightarrow Q(y)$, where $x$ is quantified over the domain of students in the class, and $y$ is quantified over the domain of all people. Since the set of students is a subset of the set of people, we are implicitly assuming that $\forall x P(x) \rightarrow Q(x)$ is true as well.

By existential instantiation on $p_1$, $P(c)$ for some student $c$. By universal instantiation on $q$, $P(c) \rightarrow Q(c)$, since the student $c$ is in the domain of people. Therefore by modus ponens, $Q(c)$. By existential generalization, $\exists x Q(x)$.

12C. Section 3.1, 12

The flaw is in the step “$n^2 \neq 3k$ for some integer $k$ implies $n \neq 3l$ for some integer $l$.” The reasoning is circular since this statement is equivalent to what we are trying to prove, and no justification for this statement is provided.

12D. Section 3.1, 26

Claim: There is an integer $n$ such that $2^n + 1$ is not prime.

Consider $n = 5$, so $2^5 + 1 = 33$. Clearly, $33 = 11 \cdot 3$, so the claim is true for $n = 5$.

1.C Handout 13

13A. Section 3.2, 2

The sum of the first $n$ even positive integers can be expressed using the following formal notation:

$$\sum_{k=1}^{n} 2k.$$ [By convention, the “empty sum” $\sum_{k=1}^{0} 2k$ is 0.]

Formally then, our claim is: $P(n)$ holds for all natural numbers $n$, where $P(n)$ is the statement $\sum_{k=1}^{n} 2k = n(n+1)$.

Proof by induction on $n$, with $P(n)$ as the induction hypothesis. Base case is $P(0)$. The sum is 0 and $0 \cdot (0+1) = 0$.

Induction step. Assume $P(n)$ is true. In the case of $P(n+1)$:

$$\sum_{k=1}^{n+1} 2k = \sum_{k=1}^{n} 2k + 2(n + 1)$$

$= n(n+1) + 2(n + 1)$ by induction hypothesis

$= (n + 1)(n + 2)$. By induction, $P(n)$ holds for all natural numbers $n$. 

13B. Section 3.2, 14
Claim: For any integer $n > 1$, $n! < n^n$.

Proof. By induction on $n$. [The induction hypothesis is $n! < n^n$.] Base case is $n = 2$; in this case $2 = 2! < 2^2 = 4$.

Induction step: Assume the claim is true for $n$. Then $(n + 1)! = (n + 1)n! < (n + 1)n^n$ by induction hypothesis. Furthermore, $(n + 1)n^n < (n + 1)(n + 1)^n = (n + 1)^{n+1}$, since $n > 1$. The claim holds for $n + 1$, therefore by induction the claim holds in general.

\[ \square \]

13C. Section 3.2, 20

Claim: For any integer $n \geq 0$, 3 divides $n^3 + 2n$.

Proof. By induction on $n$. [The induction hypothesis is 3 divides $n^3 + 2n$.] Base case is when $n = 0$, and 3 divides $0^3 + 2 \cdot 0 = 0$ trivially.

Induction step: Assume claim is true for $n$. We must check to see if 3 divides $(n + 1)^3 + 2(n + 1)$.

$(n + 1)^3 + 2(n + 1) = (n^3 + 2n) + 3n^2 + 3n + 3$. By induction hypothesis, there exists a $k$ such that $3k = n^3 + 2n$. Therefore $(n + 1)^3 + 2(n + 1) = 3k + 3n^2 + 3n + 3 = 3(k + n^2 + n + 1)$, and 3 divides $(n + 1)^3 + 2(n + 1)$. So the claim holds for $n + 1$.

\[ \square \]

13D. Section 3.2, 48

The high-level structure of the proof is legitimate, formally speaking. (Recall the second principle of mathematical induction.) The low-level reasoning in the body of the inductive step is where the logical flaw lies.

Specifically, he (tacitly) infers the equation $a^{n-1} = 1$ from the hypothesis $\forall k [0 \leq k \leq n \rightarrow a^k = 1]$, a step that is valid only if $0 \leq n - 1 \leq n$. Although the $n - 1 \leq n$ part of that implicit assumption can easily be justified, the $0 \leq n - 1$ part is unwarranted. Indeed, when $n = 0$, i.e., when we’re “proving the $P(1)$ case,” the quantity $n - 1$ is negative.