1.A  From textbook Section 1.3: 6 and 20

\(P(x, y)\) means student \(x\) has taken class \(y\).

a)  \(\exists x \exists y P(x, y)\)
There is a student in my class who has taken a class in computer science.

b)  \(\exists x \forall y P(x, y)\)
There is a student in my class who has taken all computer science classes.

c)  \(\forall x \exists y P(x, y)\)
Every student in my class has taken a class in computer science.

d)  \(\exists y \forall x P(x, y)\)
There is a computer science class every body in my class has taken.

e)  \(\forall y \exists x P(x, y)\)
For every class in the Computer Science Department, there is a student from my class who has taken it. Or:
In each computer science class there is or has been a student from my class.

f)  \(\forall x \forall y P(x, y)\)
Every student in my class has taken all computer science classes.

\(Q(x, y) \iff x + y = x - y\)

a)  \(Q(1, 1) = \text{False} \quad 2 = 1 + 1 \neq 1 - 1 = 0\)

b)  \(Q(2, 0) = \text{True} \quad 2 - 0 = 2 + 0\)

c)  \(\forall y \ Q(1, y) = \text{False}\) for example take \(y = 1\)

d)  \(\exists x \ Q(x, 2) = \text{False}\) this is equivalent to \(4 = 0\)

e)  \(\exists x \exists y \ Q(x, y) = \text{True}\) take \(x = 0\) and \(y = 1\)

f)  \(\forall x \exists y \ Q(x, y) = \text{True}\) take \(y = 0\)

g)  \(\exists y \forall x \ Q(x, y) = \text{True}\) take \(y = 0\)

h)  \(\forall y \exists x \ Q(x, y) = \text{False}\) take \(y = 1\)

i)  \(\forall x \forall y \ Q(x, y) = \text{False}\) take \(x = 2\) and \(y = 1\)

1.B  Move \(\neg\) inside

\[\neg \forall x \exists y \ (A(x, y) \rightarrow B(x, y))\]
\[\exists x \neg \exists y \ (A(x, y) \rightarrow B(x, y))\]
\[\exists x \forall y \ 
eg (A(x, y) \rightarrow B(x, y))\]
\[\exists x \forall y \ 
eg (\neg A(x, y) \lor B(x, y))\]
\[\exists x \forall y \ (A(x, y) \land \neg B(x, y))\]
\[\neg (A \lor B) \iff \neg A \land \neg B\]

1.C  Show that \(\forall x (A(x) \rightarrow B(x)) \rightarrow (\forall x A(x)) \rightarrow (\forall x B(x))\) (*)

In other words:
assuming \(\forall x (A(x) \rightarrow B(x))\) (1), show \((\forall x A(x)) \rightarrow (\forall x B(x))\) (2).

- When \((\forall x A(x))\) is false, there is nothing to do, the proposition (2) holds.
- If \((\forall x A(x))\) is true, then for every \(x\) \(A(x)\) is true, from (1) we have \(B(x)\) true, so \((\forall x B(x))\) is true.
Therefore (*) holds.
The reverse doesn’t hold, take A to be “x > 2” and B to be “x > 4”, the universe of disclosure being \( \mathbb{R} \)

2.A From Textbook Section 1.4: 12

a) \( |\emptyset| = 0 \)  
b) \( |\{\emptyset\}| = 1 \)  
c) \( |\{\emptyset, \emptyset\}| = 2 \)  
d) \( |\{\emptyset, \{\emptyset\}\}| = 3 \)

This is how Von Neumann constructed the set of integers \( \mathbb{N} \) from scratch.

2.B From Textbook Section 1.5: 12de, 38

12 d) \( (A - C) \cap (C - B) = \emptyset \)

Let’s take \( x \in (A - C) \cap (C - B) \)

- \( x \in (A - C) \) so \( x \in A \) and \( x \notin C \)
- \( x \in (C - B) \) so \( x \in C \) and \( x \notin B \)

So \( x \in C \) and \( x \notin C! \) No such \( x \) exists.

12 e) \( (B - A) \cup (C - A) = (B \cup C) - A \)

\( (B - A) \cup (C - A) = \{x \mid x \in (B - A) \lor x \in (C - A)\} = \{x \mid (x \in B \land x \notin A) \lor (x \in C \land x \notin A)\} = \{x \mid (x \in B \lor x \in C) \land x \notin A\} = (B \cup C) - A \)

38) \( U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \)

a) \( \{3, 4, 5\} \Rightarrow 0011100000 \)  
b) \( \{1, 3, 6, 10\} \Rightarrow 1010010001 \)  
c) \( \{2, 3, 4, 7, 8, 9\} \Rightarrow 0111011100 \)

3.A From Textbook Section 1.6: 16, 44, 48

16 \( S = \{-1, 0, 2, 4, 7\} \)

a) \( f(x) = 1 \)  
f\( (S) = \{1\} \)

b) \( f(x) = 2x + 1 \)  
f\( (S) = \{-1, 1, 5, 9, 15\} \)

c) \( f(S) = \lfloor x/5 \rfloor \)  
f\( (S) = \{0, 1, 2\} \)

d) \( f(S) = \lfloor (x^2 + 1)/3 \rfloor \)  
f\( (S) = \{0, 1, 5, 16\} \)

Don’t forget to take the integer part, that \( f(S) \) is a set, and don’t write duplicate numbers in sets.

44 The number of bytes needed to encode \( n \) bits is \( p = \lceil \frac{n}{8} \rceil \)

a) \( n = 4 \)  \( p = 1 \)  
b) \( n = 10 \)  \( p = 2 \)  
c) \( n = 500 \)  \( p = 63 \)  
d) \( n = 3000 \)  \( p = 375 \)

48 Graph of the function \( f(n) = 1 - n^2 \) from \( \mathbb{Z} \) to \( \mathbb{Z} \). 
See graph on the right.

3.B From Handout

\( f(x) = x^3 - 1, g(x) = x + 1 \), the set is \( \mathbb{R} \).

\( f \circ g(x) = f(g(x)) = f(x + 1) = (x + 1)^3 - 1 \)

\( f \circ g(x) = x^3 + 3x + 3x \)

Since \( \sqrt[3]{x^3 - 1} + 1 = \sqrt[3]{f(x) + 1} = x \)

\( f^{-1}(x) = \sqrt[3]{x + 1} \)