1. Reading: K. Rosen *Discrete Mathematics and Its Applications*, 4.5

2. The main message of this lecture:

   **Expected value is the exact equivalent of an "average value" intuition.**

**Example 21.1.** To enter a game one has to pay $3. Then a die is rolled, and the player gets paid the number of $$ equal to the number of points which the die comes up. Should a player accept the conditions and play this game? Imagine the game has been played 600 times (to get enough statistics). In about $1/6$ of the cases (total about 100) the die comes up 1 and then the player loses $3 - 2 = -2$, the total loss from those will be $100 \cdot (-2)$. Similarly, in $1/6$ of the cases the die comes up 2 and the whole loss is about $100 \cdot (-1)$. When the die shows 3, the player is even. The contributions of the cases when the die comes up 4, 5 or 6 are $100 \cdot 1$, $100 \cdot 2$ and $100 \cdot 3$ respectively. The estimate $G$ of the overall gain/loss is then

   \[ G = 100 \cdot (-2) + 100 \cdot (-1) + 100 \cdot 0 + 100 \cdot 1 + 100 \cdot 2 + 100 \cdot 3 = 300 \]

To get an estimate of the average gain/loss per game take

   \[ E = \frac{G}{600} = \frac{100}{600} \cdot (-2) + \frac{100}{600} \cdot (-1) + \frac{100}{600} \cdot 0 + \frac{100}{600} \cdot 1 + \frac{100}{600} \cdot 2 + \frac{100}{600} \cdot 3 = \]

   \[ = \frac{1}{6} \cdot (-2) + \frac{1}{6} \cdot (-1) + \frac{1}{6} \cdot 0 + \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 = \frac{3}{6} = 1/2 \]

The standard reading of this figure: as average, a player gains $0.5$ per game. Note that the last sum is nothing but

   \[ \sum_{i=1}^{6} \text{(probability of i)} \cdot \text{(the gain when i comes up)} \]

**Example 21.2.** A lottery ticket costs $10, a probability to win is $1/5000000$ and the winner gets $10000000$. How much a player wins/loses in average? We use the same idea:

   \[ E = \frac{1}{5000000} \cdot (10000000 - 10) + (1 - \frac{1}{5000000}) \cdot (-10) = 2 - \frac{10}{5000000} - 10 + \frac{10}{5000000} = -8 \$

The sign "-" indicates that a player loses the average of $8$ each game.

**Definition 21.3.**

A random variable $X(t)$ is a function from a sample space $S$ to the set of reals $\mathbb{R}$.

**Example 21.4.** The cost function $X(t)$ from the game 21.1 may be regarded as a random variable that maps individual outcomes from the samples space $u_1, u_2, u_3, u_4, u_5, u_6$ to the real numbers $-2, -1, 0, 1, 2, 3$ respectively. In 21.2 the sample space $S = \{W, L\}$ has probabilities $p(W) = 1/5000000$, $p(L) = 4999999/5000000$. The function $X(W) = 10000000 - 10$, $X(L) = -10$ is a random variable.

Note that a random variable is neither random nor variable!
Definition 21.5. An expected value (or expectation) of a random variable $X(t)$ is

$$E(X) = \sum_{s \in S} p(s) \cdot X(s)$$

Example 21.6. $X(s)$ is the number of Heads out of three flippings of a coin. $X(TTT) = 0$, $X(HTT) = X(TTH) = 1$, $X(HHT) = X(HTH) = X(THH) = 2$, $X(HHH) = 3$.

$$E(X) = \frac{1}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 3 =$$

$$= \frac{1}{8} \cdot 0 + 3 \cdot \frac{1}{8} \cdot 1 + 3 \cdot \frac{1}{8} \cdot 2 + \frac{1}{8} \cdot 1 = \frac{3}{2}$$

A grouping in a formula for expected value above can be generalized to a practical formula for computing the expected value:

$$E(X) = \sum_{r \in X(S)} p(X = r) \cdot r$$

Example 21.7. $X(s)$ is the sum on a pair of dice: $X(1,1) = 2, \ldots, X(6,6) = 12$. Then

$$E(X) = 2 \cdot p(X=2) + 3 \cdot p(X=3) + \ldots + 11 \cdot p(X=11) + 12 \cdot p(X=12) =$$

$$= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} = 7$$

Theorem 21.8. For $X,Y$ random variables on a space $S$

$$E(X + Y) = E(X) + E(Y), \quad E(aX) = aE(X), \quad E(aX + b) = aE(X) + b.$$  

Proof. $E(X + Y) = \sum p(s)[X(s) + Y(s)] = \sum p(s)X(s) + \sum p(s)Y(s) = E(X) + E(Y)$

$E(aX) = \sum p(s)aX(s) = a \sum p(s)X(s) = aE(X)$,

$E(b) = \sum p(s) \cdot b = b \sum p(s) = b \cdot 1 = b$,

$E(aX + b) = E(aX) + E(b) = aE(X) + b$.

Definition 21.9. Random variables $X$ and $Y$ are independent if $p(X = a \text{ and } Y = b) = p(X = a) \cdot p(Y = b)$ for any $a \in X(S), b \in Y(S)$.

Examples. Two dice, $X(i,j) = i, Y(i,j) = j$. Then $p(X = i \text{ and } X = j) = p(i,j) = 1/36$. On the other hand, $p(X = i) = p(Y = j) = 1/6$, therefore $p(X = i) \cdot p(Y = j) = 1/36$. Within the same model, $X$ and $Z = X + Y$ are not independent, since $p(X = 1 \text{ and } Y = 10) = 0$ whereas $p(X = 1)$ and $p(Z) = 1/12$.

Theorem 21.10. If $X,Y$ are independent then $E(X \cdot Y) = E(X) \cdot E(Y)$.

Definition 21.11. The variance of $X$:

$$V(X) = E(X - E(X))^2 = \sum_{s \in S} [X(s) - E(X)]^2 \cdot p(s)$$

The standard deviation $\sigma(X) = \sqrt{V(X)}$. Note that $V(X) = E(X - E(X))^2 = [X^2 - 2X \cdot E(X) + [E(X)]^2 = E(X^2) - E(2X \cdot E(X)) + [E(X)]^2 =$$

$$= E(X^2) - 2 \cdot E(X) \cdot E(X) + [E(X)]^2 = E(X^2) - [E(X)]^2.$$

Theorem 21.12. If $X_1, X_2, \ldots, X_n$ are pairwise independent, then

$$V(X_1 + \ldots + X_n) = V(X_1) + \ldots + V(X_n).$$

Example: two dice $X(i,j) = i, Y(i,j) = j$ are independent, therefore, $V(X + Y) = V(X) + V(Y)$. $V(X) = E(X^2) - [E(X)]^2 = (1^2 + 2^2 + \ldots + 6^2) \cdot (1/6) - [(1 + 2 + \ldots + 6) \cdot (1/6)]^2 = 35/12 = V(Y)$. Therefore $V(X + Y) = 35/12 + 35/12 = 35/6$

Homework assignments. (due Friday 03/16)  
21A:Rosen4.5-30; 21B:Rosen4.5-32; 21C:Rosen4.5-34; 21D:Rosen4.5-44.